This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Educ T 128.58.423

HARVARD COLLEGE
LIBRARY

GIFT OF THE
GRADUATE SCHOOL
OF EDUCATION

3 2044 097 010 441

A

KEY

TO

GREENLEAF'S ALGEBRA.

DESIGNED

FOR THE USE OF TEACHERS ONLY.

BY

BENJAMIN GREENLEAF, A. M.

BOSTON:
PUBLISHED BY ROBERT S. DAVIS & CO.
NEW YORK: D. APPLETON & CO., AND MASON BROTHERS.
PHILADELPHIA: LIPPINCOTT, GRAMBO, & CO.
And sold by the trade generally.

1858.

Educ T 128.58.423

Entered according to Act of Congress, in the year 1854, by
BENJAMIN GREENLEAF,
In the Clerk's Office of the District Court of the District of Massachusetts.

GREENLEAF'S SERIES OF MATHEMATICS.

1. PRIMARY ARITHMETIC; OR, MENTAL ARITHMETIC, upon the Inductive Plan; designed for Primary Schools. Improved electrotype edition. 72 pp.

2. INTELLECTUAL ARITHMETIC; OR, HIGHER MENTAL ARITHMETIC, upon the Inductive Plan; designed for Common Schools and Academies. Improved edition.

3. COMMON SCHOOL ARITHMETIC; OR, INTRODUCTION TO THE NATIONAL ARITHMETIC. Improved stereotype edition. 324 pp.

4. HIGHER ARITHMETIC; OR, THE NATIONAL ARITHMETIC, being a complete course of Higher Arithmetic, for advanced scholars in Common Schools and Academies. New electrotype edition, with additions and improvements. 444 pp.

5. PRACTICAL TREATISE ON ALGEBRA, for Academies and High Schools, and for advanced Students in Common Schools. Improved stereotype edition. 360 pp.

6. ELEMENTS OF GEOMETRY, for Academies and High Schools, and for Advanced Students in Common Schools. [In preparation, and will soon be published.]

COMPLETE KEYS TO THE INTRODUCTION, AND NATIONAL ARITHMETIC, AND THE PRACTICAL TREATISE ON ALGEBRA, containing Solutions and Explanations, for Teachers only. In 3 volumes.

☞ Two editions of the NATIONAL ARITHMETIC, and also of the COMMON SCHOOL ARITHMETIC, one containing the ANSWERS to the examples, and the other without them, are published. Teachers are requested to state in their orders *which edition* they prefer.

Stereotyped by
HOBART & ROBBINS,
New England Type and Stereotype Foundery,
BOSTON.

HARVARD COLLEGE LIBRARY
GIFT OF THE
GRADUATE SCHOOL OF EDUCATION
OCT 11 1930

KEY

TO

GREENLEAF'S ALGEBRA.

PRACTICAL EXAMPLES.

ARTICLE **55.** (p. 15.)

Ex. (11.) $\overline{2\times36}+\overline{3\times5\times4}-5=127.$

(12.) $\overline{5\times36\times5}-\overline{10\times6\times25}+27\times0=-600.$

(13.) $\overline{7\times36}+(5-4)\times(1-0)=253.$

(14.) $\frac{6\times25}{4}\times1-\frac{6-5}{1}+27\times36\times0=36\frac{1}{2}.$

(15.) $3\sqrt{}4+2\times6\sqrt{}(\overline{2\times6}+\overline{5-1})=54.$

(16.) $6\sqrt{}(36+0)+3\times5\times4(36-25)=696.$

(17.) $3\times36\times5+\sqrt[3]{}(16-\sqrt{(2\times6\times4+16)})-3\times0=$ 542.

(18.) $\frac{\overline{2\times5}+4}{\overline{3\times6}-4}-\frac{\sqrt{5\times5}+3\sqrt{}4+1}{\overline{2\times6}+4}=\frac{1}{4}.$

ADDITION.

ART. **59.** (p. 17.)

Ex. (6.) $=32x.$

(7.) $=35abc.$

(8.) $=14y.$

Ex. (9.) $=20mn.$

(10.) $=16h+14x$

ART. 60. (p. 19.)

Ex. (6.) $=-y$.
(7.) $=41mn$.
(8.) $=0$.

Ex. (9.) $=-5pn$.
(10.) $=0$.

Ex. (11.)

$$\begin{array}{lr} \text{Add} & 4+\ a^2x \\ \text{to} & 6-\ a^2x \\ & 3+6a^2x \\ & 15-5a^2x \\ & 3+\ a^2x \\ & 6+7a^2x \\ \hline & 37+9a^2x. \end{array}$$

Ex. (12.)

$$\begin{array}{lr} \text{Add} & 14ax-\ 6y \\ \text{to} & 7ax+\ \ y \\ & 5ax-\ 7y \\ & 9ax-11y \\ & 8ax+\ 3y \\ \hline & 43ax-20y. \end{array}$$

Ex. (13.)

$$\begin{array}{lr} \text{Add} & 3a-\ 4b+\ 6c \\ \text{to} & 7a+11b-\ 3c \\ & 8a+\ \ b-\ 7c \\ & a-11b+15c \\ \hline & 19a-\ 3b+11c. \end{array}$$

Ex. (14.)

$$\begin{array}{lr} \text{Add} & 16x^2-5y^3-16 \\ \text{to} & 3x^2+4y^3-\ 5 \\ & x^2+3y^3-37 \\ & x^2-\ y^3+\ 7 \\ & 6x^2+7y^3-11 \\ & 2x^2-3y^3-21 \\ \hline & 29x^2+5y^3-83. \end{array}$$

Ex. (15.)

$$\begin{array}{lr} \text{Add} & 5a-\ b\qquad \\ \text{to} & 3b+\ 3c \\ & 4a\qquad-\ 5c \\ & 5a-5b-\ \ c \\ & 7a\qquad-\ 6c \\ & 11a+4b-\ 7c \\ \hline & 32a+\ b-16c. \end{array}$$

ART. 63. (p. 20.)

Ex. (3.) $2x^2y+3axy-4xy^2$.

Ex. (4.) $10ax+5ax^2+8y^2+x^2+3xy+26$.

Ex. (5.)
$7ax+5a^2+7xy-169+3x^2$.

Ex. (6.) $a-1+x+y$.

Ex. (7.)

$$\begin{array}{lr} \text{Add} & 3a+b-10 \\ \text{to} & c-d-a \\ & -4c+2a-3b-7 \\ & 4x^2+5-18m \\ \hline & 4a-2b-12-3c-d+4x^2-18m. \end{array}$$

Ex. (8.)

$$\begin{array}{lr} \text{Add} & 7a-5y^3 \\ \text{to} & 8\sqrt{x}+2a \\ & 5y^3-\sqrt{x} \\ & -9a+7\sqrt{x} \\ \hline & 14\sqrt{x}. \end{array}$$

Ex. (9.)

$$\begin{array}{rl} \text{Add} & 4mn+3ab-4c \\ \text{to} & 3x\ -4ab+2mn \\ & 3m^2\ -4p \\ \hline & 6mn-ab-4c+3x+3m^2-4p. \end{array}$$

Ex. (10.)

$$\begin{array}{rl} \text{Add} & 3a^2+\ 2ab+\ 4b^2 \\ \text{to} & 5a^2-\ 8ab+\ b^2 \\ & -a^2+\ 5ab-\ b^2 \\ & 18a^2-20ab-19b^2 \\ & 14a^2-\ 3ab+20b^2 \\ \hline & 39a^2-24ab+\ 5b^2. \end{array}$$

Ex. (11.)

$$\begin{array}{rl} \text{Add} & 4x^3-5a^3-5ax^2+6a^2x \\ \text{to} & 6a^3+3x^3+4ax^2+2a^2x \\ & -17x^3+19ax^2-15a^2x \\ & 13ax^2-27a^2x+18a^3 \\ & 3a^2x-20a^3+12x^3 \\ & 31a^2x-2x^3-31ax^2-7x^3 \\ \hline & -7x^3-a^3. \end{array}$$

ART. **64.** (p. 21.)

Ex. (14.)

$$\begin{array}{rl} \text{Add} & 4ax-5my \\ \text{to} & 3dx+7ny \\ & 7mx+4my \\ \hline & (4a+3d+7m)x+(7n-m)y. \end{array}$$

Ex. (15.)

$$\begin{array}{rl} \text{Add} & 3hz-5x \\ \text{to} & 4mz+nx \\ & 5az-4px \\ \hline & (3h+4m+5a)z+(n-5-4p)x. \end{array}$$

SUBTRACTION.

ART. **70.** (p. 24.)

Ex. (10.)

$$\begin{array}{rl} \text{From} & 3a-5b+6h-\ d \\ \text{Take} & a+\ b\qquad -7d \\ \hline & 2a-6b+6h+6d. \end{array}$$

Ex. (11.)

$$\begin{array}{rl} \text{From} & 31x^2-3y^2+ab \\ \text{Take} & 17x^2+5y^2-4ab+7 \\ \hline & 14x^2-8y^2+5ab-7. \end{array}$$

Ex. (12.)

$$\begin{array}{rl} \text{From} & 5f+14b-\ 9d \\ \text{Take} & -3f+\ 7b-15d \\ \hline & 8f+\ 7b+\ 6d. \end{array}$$

Ex. (13.)

$$\begin{array}{rl} \text{From} & 11a-\ 7b+\ c \\ \text{Take} & a+\ 7b-3c+11 \\ \hline & 10a-14b+4c-11. \end{array}$$

Ex. (14.)

$$\begin{array}{ll} \text{From} & m^2+3n^3 \\ \text{Take} & -4m^2-6n^3+71x \\ \hline & 5m^2+9n^3-71x. \end{array}$$

Ex. (15.)

$$\begin{array}{ll} \text{From} & 31a-15x-7 \\ \text{Take} & 2a-25x+y^2 \\ \hline & 29a+10x-y^2-7. \end{array}$$

Ex. (16.)

$$\begin{array}{ll} \text{From} & abc^2-\ xy^3 \\ \text{Take} & -6abc^2+3xy^3-7h \\ \hline & 7abc^2-4xy^3+7h. \end{array}$$

Ex. (17.)

$$\begin{array}{ll} \text{From} & 11ch^2-5 \\ \text{Take} & 5ch^2-5+47x \\ \hline & 6ch^2-47x. \end{array}$$

Ex. (18.)

$$\begin{array}{ll} \text{From} & mn^2+kt \\ \text{Take} & -7mn^2+48x-y^2 \\ \hline & 8mn^2+kt-48x+y^2. \end{array}$$

Ex. (19.)

$$\begin{array}{ll} \text{From} & 47abh-37+96y^2 \\ \text{Take} & 7abh \\ \hline & 40abh-37+96y^2. \end{array}$$

Ex. (20.)

$$\begin{array}{ll} \text{From} & 8x^2y^3+17 \\ \text{Take} & 7x^2y^3+hm \\ \hline & x^2y^3-hm+17. \end{array}$$

Ex. (21.)

$$\begin{array}{ll} \text{From} & 11b^2 \\ \text{Take} & 5b^2-3c+59m \\ \hline & 6b^2+3c-59m. \end{array}$$

Ex. (22.)

$$\begin{array}{ll} \text{From} & 6a+3b-5c+1 \\ \text{Take} & 6a-3b-5c \\ \hline & 6b+1. \end{array}$$

Ex. (23.)

$$\begin{array}{ll} \text{From} & m^2 \\ \text{Take} & 41x^2+7y^3+abc \\ \hline & m^2-41x^2-7y^3-abc. \end{array}$$

Ex. (24.)

$$\begin{array}{ll} \text{From} & -17x^2+14y-a+b \\ \text{Take} & x^2 \\ \hline & -18x^2+14y-a+b. \end{array}$$

Ex. (25.)

$$\begin{array}{ll} \text{From} & a+b \\ \text{Take} & a-b \\ \hline & +2b. \end{array}$$

Ex. (26.)

$$\begin{array}{ll} \text{From} & 9xz \\ \text{Take} & xz-7h-5m^3+7 \\ \hline & 8xz+7h+5m^3-7. \end{array}$$

Ex. (27.)

$$\begin{array}{ll} \text{From} & 11hm+8n^2 \\ \text{Take} & x^2-y^2 \\ \hline & 11hm+8n^2-x^2+y^2. \end{array}$$

Ex. (28.)

$$\begin{array}{rr}
\text{From} & a+b \\
\text{Take} & a-b \\ \cline{2-2}
 & 2b \\
\text{and} & a-\ b \\ \cline{2-2}
 & -a+3b \\
\text{and} & -a+\ b \\ \cline{2-2}
 & +2b.
\end{array}$$

Ex. (29.)

$$\begin{array}{rr}
\text{From} & a-b-c \\
\text{Take} & -a+b+c \\ \cline{2-2}
 & 2a-2b-2c \\
\text{and} & a-\ b+\ c \\ \cline{2-2}
 & a-\ b-3c.
\end{array}$$

ART. 72. (p. 26.)

Ex. (1.)

$$\begin{array}{rl}
\text{To} & 1-2x+3x^2 \\
\text{Add} & 3+2x-\ x^2 \\ \cline{2-2}
 & 4\qquad\ +2x^2.
\end{array}$$

Ex. (2.)

$$\begin{array}{rr}
\text{To} & 5a-4b+3c \\
\text{Add} & -3a+2b-\ c \\ \cline{2-2}
 & 2a-2b+2c.
\end{array}$$

Ex. (3.)

$$\begin{array}{llll}
a-b-c & & & \\
 & b+c-d & & \\
 & & d-e+f & \\
 & & & e-f-g \\ \hline
a & & & -g.
\end{array}$$

Ex. (4.)

$$\begin{array}{r}
a-b+c \\
-a+b+c \\
-a+b+c \\ \hline
-a+b+3c.
\end{array}$$

Ex. (5.)

$$\begin{array}{rrr}
 & \multicolumn{2}{r}{x^2+2xy+y^2} \\
 & \multicolumn{2}{r}{-x^2+2xy-y^2} \\ \cline{2-3}
\text{Take} & 2x^2 & +2y^2 \\
\text{From} & 3x^2 & +3y^2 \\ \cline{2-3}
 & x^2 & +\ y^2.
\end{array}$$

Ex. (6.)

$$\begin{array}{ll}
\text{From} & 6x^2+2y^2-3x^2-y^2 \\
\text{Take} & 2x^2+4y^2-4x^2+y^2 \\ \hline
\multicolumn{2}{l}{4x^2-2y^2+x^2-2y^2=5x^2-4y^2.}
\end{array}$$

MULTIPLICATION.

ART. 83. (p. 32.)

Ex. (9.)

$$\begin{array}{r}
5a^2x-7y+4x^3-3b^3 \\
4ay^2 \\ \hline
20a^3xy^2-28ay^3+16ax^3y^2-12ab^3y^2.
\end{array}$$

Ex. (10.)

$$\begin{array}{r} 7a^2b^3+4am^2-6y \\ 4a^5m^3 \\ \hline 28a^7b^3m^3+16a^6m^5-24a^5m^3y. \end{array}$$

Ex. (11.)

$$\begin{array}{r} 4a^2b^3-6a^3c+c^2 \\ -5a^2 \\ \hline -20a^4b^3+30a^5c-5a^2c^2. \end{array}$$

Ex. (12.)

$$\begin{array}{r} -ab^2-3x^3-14m^{-2} \\ -am \\ \hline a^2b^2m+3amx^3+14am^{-1}. \end{array}$$

ART. **86.** (p. 34.)

Ex. (10.)

$$\begin{array}{l} 1+x \\ 1+x^4 \\ \hline 1+x \\ \quad x^4+x^5 \\ \hline 1+x+x^4+x^5 \\ 1-x+x^2-x^3 \\ \hline 1+x+x^4+x^5 \\ \quad -x-x^2-x^5-x^6 \\ \qquad +x^2+x^3+x^6+x^7 \\ \qquad\qquad -x^3-x^4-x^7-x^8 \\ \hline 1 \qquad\qquad\qquad\qquad -x^8. \end{array}$$

Ex. (11.)

$$\begin{array}{l} a+2x \\ a-3x \\ \hline a^2+2ax \\ \quad -3ax-6x^2 \\ \hline a^2-\ ax-6x^2 \\ a+4x \\ \hline a^3-\ a^2x-\ 6ax^2 \\ \qquad 4a^2x-\ 4ax^2-24x^3 \\ \hline a^3+3a^2x-10ax^2-24x^3. \end{array}$$

Ex. (12.)

$$\begin{array}{l} 3a-\ x \\ 2a+4x \\ \hline 6a^2-\ 2ax \\ \quad +12ax-4x^2 \\ \hline 6a^2+10ax-4x^2 \\ 4a-\ 2x \\ \hline 24a^3+40a^2x-16ax^2 \\ \qquad -12a^2x-20ax^2+8x^3 \\ \hline 24a^3+28a^2x-36ax^2+8x^3. \end{array}$$

Ex. (13.)

$$\begin{array}{l} 3x^2-2xy-y^2 \\ 2x-4y \\ \hline 6x^3-\ 4x^2y-2xy^2 \\ \qquad -12x^2y+8xy^2+4y^3 \\ \hline 6x^3-16x^2y+6xy^2+4y^3. \end{array}$$

Ex. (14.)

$$\begin{array}{l} x^2+2x+1 \\ x^2-2x+3 \\ \hline x^4+2x^3+\ x^2 \\ \quad -2x^3-4x^2-2x \\ \qquad\qquad 3x^2+6x+3 \\ \hline x^4 \qquad\qquad\quad +4x+3. \end{array}$$

Ex. (15.)

$$\begin{array}{l} a+b-c \\ a-b+c \\ \hline a^2+ab-ac \\ \quad -ab-b^2+bc \\ \qquad\qquad ac+bc-c^2 \\ \hline a^2 \qquad -b^2+2bc-c^2. \end{array}$$

Ex. (16.)

$$\begin{array}{l} 3a-2b \\ -2a+4b \\ \hline -6a^2+4ab \\ \qquad +12ab-8b^2 \\ \hline -6a^2+16ab-8b^2. \end{array}$$

Ex. (17.)

$$\begin{array}{l} 5a^2-3ab+4b^2 \\ 6a-5b \\ \hline 30a^3-18a^2b+24ab^2 \\ \qquad -25a^2b+15ab^2-20b^3 \\ \hline 30a^3-43a^2b+39ab^2-20b^3. \end{array}$$

Ex. (18.)

$$\begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad -b^3. \end{array}$$

Ex. (19.)

$$\begin{array}{l} a^4-x^4 \\ a^4-x^4 \\ \hline a^8-a^4x^4 \\ \quad -a^4x^4+x^8 \\ \hline a^8-2a^4x^4+x^8. \end{array}$$

Ex. (20.)

$$\begin{array}{l} 2x^2-3xy+6 \\ 3x^2+3xy-5 \\ \hline 6x^4-9x^3y+18x^2 \\ \quad +6x^3y-9x^2y^2+18xy \\ \qquad\qquad -10x^2+15xy-30 \\ \hline 6x^4-3x^3y+8x^2-9x^2y^2+33xy-30 \end{array}$$

Ex. (21.)

$$\begin{array}{l} 5a^2-4ax+3x^2 \\ 2a^2-3ax-4x^2 \\ \hline 10a^4-8a^3x+6a^2x^2 \\ \quad -15a^3x+12a^2x^2-9ax^3 \\ \qquad\qquad -20a^2x^2+16ax^3-12x^4 \\ \hline 10a^4-23a^3x-2a^2x^2+7ax^3-12x^4. \end{array}$$

Ex. (22.)

$$\begin{array}{l} 2a^2-3ax+4x^2 \\ 5a^2-6ax-2x^2 \\ \hline 10a^4-15a^3x+20a^2x^2 \\ \quad -12a^3x+18a^2x^2-24ax^3 \\ \qquad\qquad -4a^2x^2+6ax^3-8x^4 \\ \hline 10a^4-27a^3x+34a^2x^2-18ax^3-8x^4 \end{array}$$

Ex. (23.)

$$\begin{array}{r}
a^3-3a^2+3a-1 \\
a^2-2a+1 \\
\hline
a^5-3a^4+3a^3-a^2 \\
-2a^4+6a^3-6a^2+2a \\
a^3-3a^2+3a-1 \\
\hline
a^5-5a^4+10a^3-10a^2+5a-1.
\end{array}$$

Ex. (24.)

$$\begin{array}{l}
a^m-a^n \\
2a-a^n \\
\hline
2a^{m+1}-2a^{n+1}-a^{m+n}+a^{2n}.
\end{array}$$

Ex. (25.)

$$\begin{array}{l}
a^4-a^3x+a^2x^2-ax^3+x^4 \\
a+x \\
\hline
a^5-a^4x+a^3x^2-a^2x^3+ax^4 \\
\qquad a^4x-a^3x^2+a^2x^3-ax^4+x^5 \\
\hline
a^5 \qquad\qquad\qquad\qquad\qquad\qquad +x^5.
\end{array}$$

Art. 87. (p. 36.)

Ex. (5.)

$$\begin{array}{r}
5+0+0-3+1 \\
2+1 \\
\hline
10+0+0-6+2 \\
5+0+0-3+1 \\
\hline
10+5+0-6-1+1 \\
10a^9+5a^8-6a^6-a^5+a^4.
\end{array}$$

Ex. (6.)

$$\begin{array}{r}
3+0-2-2 \\
1+0-3 \\
\hline
3+0-2-2 \\
-9-0+6+6 \\
\hline
3+0-11-2+6+6 \\
3x^5-11x^3-2x^2+6x+6.
\end{array}$$

Ex. (7.)

$$\begin{array}{r}
1+1-3 \\
1+0-1 \\
\hline
1+1-3 \\
-1-1+3 \\
\hline
1+1-4-1+3 \\
y^5+y^4-4y^3-y^2+3y.
\end{array}$$

Ex. (8.)

$$\begin{array}{r}
1+1+1+1+1+1 \\
1-1 \\
\hline
1+1+1+1+1+1 \\
-1-1-1-1-1-1 \\
\hline
1 \qquad\qquad\qquad\qquad -1 \\
x^6 \qquad\qquad\qquad\qquad -1.
\end{array}$$

Ex. (9.)

$$\begin{array}{r}
1-2+4 \\
1+2+4 \\
\hline
1-2+4 \\
2-4+8 \\
+4-8+16 \\
\hline
1+0+4+0+16= \\
a^4+4a^2b^2+16b^4.
\end{array}$$

Ex. (10.)

$$\begin{array}{r}
3+3+3+3+3 \\
7-7 \\
\hline
21+21+21+21+21 \\
-21-21-21-21-21 \\
\hline
21+0+0+0+0-21= \\
21a^5-21b^5.
\end{array}$$

Ex. (11.)

$$\begin{array}{l} 1+1+1+1 \\ 1-1 \\ \hline 1+1+1+1 \\ \quad -1-1-1-1 \\ \hline 1+0+0+0-1= \\ x^4-y^4. \end{array}$$

DIVISION.

Art. 89. (p. 38.)

Ex. (6.) $2a$.	(7.) $3mn$.	(8.) 17.	
Ex. (9.) $7rs$.	(10.) $5hm$.		

Art. 91. (p. 39.)

Ex. (5.) $15y^3+30x^3y^7$.
(6.) $ax^3z^3-2z+3y$.
Ex. (7.) $mq+pm-p^2mc$.
(8.) $4tx-8t^2+z$.

Art. 94. (p. 42.)

Ex. (7.)

$$x\Big(\frac{a-x}{1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4}+\frac{x^5}{a^5}}, \&c.$$

$$\begin{array}{l} \underline{a-x} \\ \quad x \\ \quad \underline{x-\frac{x^2}{a}} \\ \qquad \frac{x^2}{a} \\ \qquad \underline{\frac{x^2}{a}-\frac{x^5}{a^2}} \\ \qquad\qquad \frac{x^3}{a^2} \end{array}$$

$$\underline{\frac{x^3}{a^2}\quad\frac{x^4}{a^3}}$$

$$\frac{x^4}{a^3}$$

$$\underline{\frac{x^4}{a^3}\quad\frac{x^5}{a^4}}$$

$$\frac{x^5}{a^4}$$

$$\underline{\frac{x^5}{a^4}\quad\frac{x^6}{a^5}}$$

Ex. (8.) $a^2-2ax+x^2\left(\dfrac{a-x}{a-x}\right.$.

$$\underline{a^2-ax}$$

$$-ax+x^2$$

$$\underline{-ax+x^2}$$

Ex. (9.) $a^3-3a^2b+3ab^2-b^3\left(\dfrac{a-b}{a^2-2ab+b^2}\right.$.

$$\underline{a^3-\ a^2b}$$

$$-2a^2b+3ab^2$$

$$\underline{-2a^2b+2ab^2}$$

$$ab^2-b^3$$

$$\underline{ab^2-b^3}$$

Ex. (10.) $8a^3-4a^2b-6ab^2+3b^3\left(\dfrac{2a-b}{4a^2-3b^2}\right.$

$$\underline{8a^3-4a^2b}$$

$$-6ab^2+3b^3$$

$$\underline{-6ab^2+3b^3}$$

Ex. (11.) $3b^3+3ab^2-4a^2b-4a^3\left(\dfrac{a+b}{-4a^2+3b^2}\right.$.

$$\underline{3b^3+3ab^2}$$

$$-4a^2b-4a^3$$

$$\underline{-4a^2b-4a^3}$$

Ex. (12.)

$$\begin{array}{l}
2a^2x^2-5ax+2\left(\dfrac{2ax-1}{ax-2}\right. \\
\underline{2a^2x^2-\ ax\qquad} \\
\qquad\quad -4ax+2 \\
\qquad\quad \underline{-4ax+2}
\end{array}$$

Ex. (13.)

$$\begin{array}{l}
21a^5-21b^5\left(\dfrac{7a-7b}{3a^4+3a^3b+3a^2b^2+3ab^3+3b^4}\right. \\
\underline{21a^5-21a^4b} \\
\qquad\quad 21a^4b \\
\qquad\quad \underline{21a^4b-21a^3b^2} \\
\qquad\qquad\qquad\quad 21a^3b^2 \\
\qquad\qquad\qquad\quad \underline{21a^3b^2-21a^2b^3} \\
\qquad\qquad\qquad\qquad\qquad\quad 21a^2b^3 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{21a^2b^3-21ab^4\qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 21ab^4-21b^5 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{21ab^4-21b^5}
\end{array}$$

Ex. (14.)

$$\begin{array}{l}
x^4-y^4+2y^2z^2-z^4\left(\dfrac{x^2+y^2-z^2}{x^2-y^2+z^2}\right. \\
\underline{x^4+x^2y^2-x^2z^2\qquad} \\
\quad -\text{[illegible]}y^2-y^4+x^2z^2 \\
\quad \underline{-x^2y^2-y^4+y^2z^2\qquad\qquad\quad} \\
\qquad\qquad\qquad\quad x^2z^2+y^2z^2-z^4 \\
\qquad\qquad\qquad\quad \underline{x^2z^2+y^2z^2-z^4}
\end{array}$$

Ex. (15.)

$$\begin{array}{l}
1+a\left(\dfrac{1-a}{1+2a+2a^2+2a^3+2a^4}\right., \ \&\text{c.} \\
\underline{1-a} \\
\quad 2a \\
\quad \underline{2a-2a^2} \\
\qquad\quad 2a^2 \\
\qquad\quad \underline{2a^2-2a^3} \\
\qquad\qquad\quad 2a^3 \\
\qquad\qquad\quad \underline{2a^3-2a^4} \\
\qquad\qquad\qquad\quad 2a^4 \\
\qquad\qquad\qquad\quad \underline{2a^4-2a^5}
\end{array}$$

Ex. (16.)

$$\begin{array}{l}
8x^2-15y^2+23yz-2xy-8xz-6z^2\left(\dfrac{2x-3y+z}{4x+5y-6z}\right. \\
\underline{8x^2 \qquad\qquad\qquad -12xy+4xz} \\
\quad -15y^2+23yz+10xy-12xz \\
\quad \underline{-15y^2+5yz+10xy \qquad\qquad\quad} \\
\qquad\qquad 18yz \qquad -12xz-6z^2 \\
\qquad\qquad \underline{18yz \qquad -12xz-6z^2}
\end{array}$$

Ex. (17.)

$$\begin{array}{l}
6x^4-96\left(\dfrac{3x-6}{2x^3+4x^2+8x+16}\right. \\
\underline{6x^4-12x^3} \\
\quad 12x^3-96 \\
\quad \underline{12x^3-24x^2} \\
\qquad 24x^2-96 \\
\qquad \underline{24x^2-48x} \\
\qquad\quad 48x-96 \\
\qquad\quad \underline{48x-96}
\end{array}$$

Ex. (18.)

$$\begin{array}{l}
a^8+a^6b^2+a^4b^4+a^2b^6+b^8\left(\dfrac{a^4+a^3b+a^2b^2+ab^3+b^4}{a^4-a^3b+a^2b^2-ab^3+b^4}\right. \\
\underline{a^8+a^7b+a^6b^2+a^5b^3+a^4b^4} \\
\quad -a^7b-a^5b^3+a^2b^6+b^8 \\
\quad \underline{-a^7b-a^6b^2-a^5b^3-a^4b^4-a^3b^5} \\
\qquad a^6b^2+a^4b^4+a^3b^5+a^2b^6+b^8 \\
\qquad \underline{a^6b^2+a^5b^3+a^4b^4+a^3b^5+a^2b^6} \\
\qquad\quad -a^5b^3 \qquad\qquad +b^8 \\
\qquad\quad \underline{-a^5b^3-a^4b^4-a^3b^5-a^2b^6-ab^7} \\
\qquad\qquad a^4b^4+a^3b^5+a^2b^6+ab^7+b^8 \\
\qquad\qquad \underline{a^4b^4+a^3b^5+a^2b^6+ab^7+b^8}
\end{array}$$

ART. 95. (p. 44.)

Ex. (4.)

$$1-0-0-0-0-0-0-1\left(\frac{1+1}{1-1+1-1+1-1+1-1}\right.=$$

$$1-a+a^2-a^3+a^4-a^5+a^6-a^7.$$

$$\begin{array}{l}
1+1\\
\hline
-1\\
-1-1\\
\hline
1\\
1+1\\
\hline
-1\\
-1-1\\
\hline
1\\
1+1\\
\hline
-1\\
-1-1\\
\hline
1\\
1+1\\
\hline
-1-1\\
-1-1\\
\hline
\end{array}$$

Ex. (5.)

$$3+3-4-4\left(\frac{1+1}{3-0-4}\right.=$$

$$3y^2-4x^2.$$

$$\begin{array}{l}
3+3\\
\hline
-4-4\\
-4-4\\
\hline
\end{array}$$

Ex. (6.)

$$1-3-8+18-8\left(\frac{1+2-2}{1-5+4}\right.=$$

$$a^2-5ab+4b^2.$$

$$\begin{array}{l}
1+2-2\\
\hline
-5-6+18\\
-5-10+10\\
\hline
4+8-8\\
4+8-8\\
\hline
\end{array}$$

Ex. (7.)

$$1-5+10-10+5-1\left(\frac{1-2+1}{1-3+3-1}\right.=$$

$$m^3-3m^2n+3mn^2-n^3.$$

$$\begin{array}{l}
1-2+1\\
\hline
-3+9-10\\
-3+6-3\\
\hline
3-7+5\\
3-6+3\\
\hline
-1+2-1\\
-1+2-1\\
\hline
\end{array}$$

Ex. (8.)

$$\begin{array}{l}
1+1-1-1\Big(\dfrac{1+1}{1-0-1}= \\
\underline{1+1 \qquad\quad} \quad a^{n+1}-b^{n+1}. \\
\qquad -1-1 \\
\qquad \underline{-1-1}
\end{array}$$

QUESTIONS FOR EXERCISE. (p. 45.)

Ex. (1.)

$$\begin{array}{r}
12a+\ 5c+17d+13b \\
8a+\ 8c+15d+12b \\
15a+11c+10d+23b \\
3a+18c+\ 4d+20b \\
\hline
38a+42c+46d+68b.
\end{array}$$

Ex. (2.)

$$\begin{array}{l}
5a+3b-4c \\
2a-5b+6c+2d+3e \\
\ a-4b-2c \qquad\ -6e \\
7a+4b-3c \\
\hline
15a-2b-3c+2d-3e.
\end{array}$$

Ex. (3.)

$$\begin{array}{r}
3a^2+\ 2ab+\ 4b^2 \\
5a^2-\ 8ab+\ 6b^2 \\
-4a^2+\ 5ab-\ \ b^2 \\
18a^2-20ab-19b^2 \\
14a^2-\ 3ab+20b^2 \\
-36a^2+24ab-10b^2 \\
\hline
0\ +\ 0\ +\ 0.
\end{array}$$

Ex. (4.)

$$\begin{array}{r}
5a^2b-17a^3bc-15b^2c^4+5- \\
-4a^2b+\ 8a^3bc-10b^2c^4-4 \\
-3a^2b-\ 3a^3bc+20b^2c^4-3 \\
2a^2b+12a^3bc+\ 5b^2c^4+2 \\
\hline
0\quad +\ 0\qquad +\ 0\quad +0.
\end{array}$$

Ex. (5.)

$$\begin{array}{r}
a+\ b+\ c+\ d \\
a+\ b+\ c-2d \\
a+\ b-2c+\ d \\
a-3b+\ c+\ d \\
-a+\ b+\ c+\ d \\
a-\ b-2c-2d \\
\hline
4a. \qquad\qquad\quad
\end{array}$$

Ex. (6.)

$$\begin{array}{l}
x^2+2x+1 \\
\underline{x^2-2x+3} \\
x^4+2x^3+\ x^2 \\
\quad -2x^3-4x^2-2x \\
\qquad\qquad\quad 3x^2+6x+3 \\
\hline
x^4 \qquad\qquad\qquad +4x+3.
\end{array}$$

Ex. (7.)

$$\begin{array}{l}
1-x+x^2-x^3 \\
\underline{1+x} \\
1-x+x^2-x^3 \\
\quad\ \ x-x^2+x^3-x^4 \\
\hline
1 \qquad\qquad\quad -x^4.
\end{array}$$

Ex. (8.)

$$\begin{array}{l}
1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7\\
1+2x+x^2\\ \hline
1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7\\
\quad 2x-4x^2+6x^3-8x^4+10x^5-12x^6+14x^7-16x^8\\
\qquad\quad x^2-2x^3+3x^4-4x^5+5x^6-6x^7+7x^8-8x^9\\ \hline
1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -9x^8-8x^9.
\end{array}$$

Ex. (9.)

$$\begin{array}{l}
a+b\\
a-b\\ \hline
a^2+ab\\
\quad -ab-b^2\\ \hline
a^2-b^2\\
a^2+ab+b^2\\ \hline
a^4-a^2b^2\\
\qquad a^3b-ab^3\\
\qquad\qquad a^2b^2-b^4\\ \hline
a^4+a^3b-ab^3-b^4\\
a^2-ab+b^2\\ \hline
a^6+a^5b-a^3b^3-a^2b^4\\
\quad -a^5b-a^4b^2+a^2b^4+ab^5\\
\qquad\qquad a^4b^2+a^3b^3-ab^5-b^6\\ \hline
a^6 \qquad\qquad\qquad\qquad\qquad -b^6.
\end{array}$$

Ex. (10.)

$$\begin{array}{l}
x^3+3ax^2+3a^2x+a^3\\
x^3-3ax^2+3a^2x-a^3\\ \hline
x^6+3ax^5+3a^2x^4+a^3x^3\\
\quad -3ax^5-9a^2x^4-9a^3x^3-3a^4x^2\\
\qquad\qquad 3a^2x^4+9a^3x^3+9a^4x^2+3a^5x\\
\qquad\qquad\qquad -a^3x^3-3a^4x^2-3a^5x-a^6\\ \hline
x^6 \qquad -3a^2x^4 \qquad\qquad +3a^4x^2 \qquad\qquad -a^6.
\end{array}$$

Ex. (11.)

$$\begin{array}{l}
a^{m-1}+b^{m-1}\\
a^{n+1}-b^{n+1}\\ \hline
a^{m+n}+a^{n+1}b^{m-1}\\
\qquad -a^{m-1}b^{n+1}-b^{m+n}\\ \hline
a^{m+n}+a^{n+1}b^{m-1}-a^{m-1}b^{n+1}-b^{m+n}.
\end{array}$$

Ex. (12.)

$$\begin{array}{l} x^5-a^5\Big(\dfrac{x-a}{x^4+ax^3+a^2x^2+a^3x+a^4} \\ \underline{x^5-ax^4} \\ \quad ax^4 \\ \quad \underline{ax^4-a^2x^3} \\ \qquad a^2x^3 \\ \qquad \underline{a^2x^3-a^3x^2} \\ \qquad\quad a^3x^2 \\ \qquad\quad \underline{a^3x^2-a^4x} \\ \qquad\qquad a^4x-a^5 \\ \qquad\qquad \underline{a^4x-a^5} \end{array}$$

Ex. (13.)

$$\begin{array}{l} x^4-9x^2-6xy-y^2\Big(\dfrac{x^2+3x+y}{x^2-3x-y}. \\ \underline{x^4+3x^3+\ x^2y} \\ \quad -3x^3-9x^2-\ x^2y \\ \quad \underline{-3x^3-9x^2-3xy} \\ \qquad\qquad -x^2y-3xy-y^2 \\ \qquad\qquad \underline{-x^2y-3xy-y^2} \end{array}$$

Ex. (14.)

$$\begin{array}{l} x^4-4x^3+6x^2-4x+1\Big(\dfrac{x^2-2x+1}{x^2-2x+1}. \\ \underline{x^4-2x^3+\ x^2} \\ \quad -2x^3+5x^2-4x \\ \quad \underline{-2x^3+4x^2-2x} \\ \qquad\qquad x^2-2x+1 \\ \qquad\qquad \underline{x^2-2x+1} \end{array}$$

$$\begin{array}{l} x^4-2a^2x^2+16a^3x-15a^4\Big(\dfrac{x^2+2ax-3a^2}{x^2-2ax+5a^2}. \\ \underline{x^4+2ax^3-\ 3a^2x^2} \\ \quad -2ax^3+\ a^2x^2+16a^3x \\ \quad \underline{-2ax^3-4a^2x^2+\ 6a^3x} \\ \qquad\qquad 5a^2x^2+10a^3x-15a^4 \\ \qquad\qquad \underline{5a^2x^2+10a^3x-15a^4} \end{array}$$

From	$x^2-2ax+5a^2$
Take	x^2-2x+1
Ans.	$2x-2ax-1+5a^2$.

Ex. (15.)

$$\begin{array}{l}
x^6-16a^3x^3+64a^6\left(\begin{array}{l} x-2a \\ \hline x^5+2ax^4+4a^2x^3-8a^3x^2-16a^4x-32a^5 \end{array}\right. \\
\underline{x^6-\ 2ax^5} \\
\qquad 2ax^5 \\
\qquad \underline{2ax^5-4a^2x^4} \\
\qquad\qquad 4a^2x^4-16a^3x^3 \\
\qquad\qquad \underline{4a^2x^4-\ 8a^3x^3} \\
\qquad\qquad\qquad -8a^3x^3 \\
\qquad\qquad\qquad \underline{-8a^3x^3+16a^4x^2} \\
\qquad\qquad\qquad\qquad -16a^4x^2 \\
\qquad\qquad\qquad\qquad \underline{-16a^4x^2+32a^5x} \\
\qquad\qquad\qquad\qquad\qquad -32a^5x+64a^6 \\
\qquad\qquad\qquad\qquad\qquad \underline{-32a^5x+64a^6}
\end{array}$$

FRACTIONS.

ART. 115. (p. 49.)

Ex. (4.) First Division.

$$\begin{array}{l}
x^3-a^3\left(\begin{array}{l} x^2-a^2 \\ \hline x+a \end{array}\right. . \\
\underline{x^3-a^2x\qquad\qquad} \\
\qquad a^2x-a^3\left(\begin{array}{l} a^2 \\ \hline x-a \end{array}\right. . \\
\qquad \underline{a^2x-a^3}
\end{array}$$

Second Division.

$$\begin{array}{l}
x^2-a^2\left(\begin{array}{l} x-a \\ \hline x+a \end{array}\right. . \\
\underline{x^2-ax\quad} \\
\qquad ax-a^2 \\
\qquad \underline{ax-a^2}
\end{array}$$

Hence the greatest common divisor is $x-a$.

Ex. (5.) $x^2-1(ax+a$.

Divide both terms of the divisor by a, and we have $\overline{ax+a}\div a=x+1$.

Then,

$$\begin{array}{l}
x^2-1\left(\begin{array}{l} x+1 \\ \hline x-1 \end{array}\right. . \\
\underline{x^2+x\quad} \\
\quad -x-1 \\
\quad \underline{-x-1}
\end{array}$$

Thus we have $x+1$ for the common divisor.

Ex. (6.) First Division.

$$\begin{array}{l}
y^4-x^4\left(\dfrac{y^3-y^2x-yx^2+x^3}{y+x}\right. . \\
\underline{y^4-y^3x-y^2x^2+yx^3} \\
\quad y^3x+y^2x^2-yx^3-x^4 \\
\quad \underline{y^3x-y^2x^2-yx^3+x^4} \\
\text{Divide by } 2x^2\,)\ \underline{2y^2x^2\quad -2x^4} \\
\qquad\qquad y^2-x^2.
\end{array}$$

Second Division.

$$\begin{array}{l}
y^3-y^2x-yx^2+x^3\left(\dfrac{y^2-x^2}{y-x}\right. . \\
\underline{y^3\qquad -yx^2} \\
\quad -y^2x\qquad +x^3 \\
\quad \underline{-y^2x\qquad +x^3}
\end{array}$$

Hence the greatest common divisor is y^2-x^2.

Ex. (7.)

$$\begin{array}{l}
a^4-x^4\left(\dfrac{a^3-a^2x+ax^2-x^3}{a+x}\right. . \\
\underline{a^4-a^3x+a^2x^2-ax^3} \\
\quad a^3x-a^2x^2+ax^3-x^4 \\
\quad \underline{a^3x-a^2x^2+ax^3-x^4}
\end{array}$$

Hence the greatest common divisor is $a^3-a^2x+ax^2-x^3$.

Ex. (8.) First Division.

$$\begin{array}{l}
a^5+a^3x^2\left(\dfrac{a^4-x^4}{a}\right. . \\
\underline{a^5-ax^4} \\
\quad ax^4+a^3x^2\left(\dfrac{ax^2}{x^2+a^2}\right., \text{ or } a^2+x^2. \\
\quad \underline{ax^4+a^3x^2}
\end{array}$$

Second Division.

$$\begin{array}{l}
a^4-x^4\,(x^2+a^2, \text{ or } a^2+x^2 \\
\underline{a^4+a^2x^2} \\
\quad -a^2x^2-x^4 \\
\quad \underline{-a^2x^2-x^4}
\end{array}$$

Hence the greatest common divisor is a^2+x^2.

ART. **117.** (p. 51.)

Ex. (11.) $\dfrac{6a^2+5ax-6x^2}{6a^2+13ax+6x^2}$.

$$\begin{array}{l}
6a^2+5ax-6x^2\,)\ 6a^2+13ax+6x^2\ (\,1. \\
\qquad\qquad\qquad\ \ \underline{6a^2+\ 5ax-6x^2} \\
\qquad\qquad\qquad\quad 4x\,)\ 8ax+12x^2 \\
\hline
\text{Common measure,}\quad 2a+3x\,)\ 6a^3+5ax-6x^2\ (\,3a-2x. \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{6a^2+9ax} \\
\qquad\qquad\qquad\qquad\qquad\qquad -4ax-6x^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-4ax-6x^2}
\end{array}$$

$$2a+3x\Big(\frac{6a^2+5ax-6x^2}{6a+13ax+6x^2}=\frac{3a-2x}{3a+2x}.$$ *Ans.*

Ex. (12.) $\frac{a^2-x^2}{a^4-x^4}.$

$$\begin{array}{l}
a^2-x^2\;)\;a^4-x^4\;(\;a^2+x^2.\\
\qquad\quad\;\; \underline{a^4-a^2x^2}\\
\qquad\qquad\quad a^2x^2-x^4\\
\qquad\qquad\quad \underline{a^2x^2-x^4}
\end{array}$$

$$a^2-x^2\Big)\frac{a^2-x^2}{a^4-x^4}=\frac{1}{a^2+x^2}.$$ *Ans.*

Ex. (13.) $\frac{x^6-y^6}{x^4-y^4}.$

$$\begin{array}{l}
x^4-y^4\;)\;x^6-y^6\;(\;x^2\\
\qquad\qquad \underline{x^6-x^2y^4}\\
\qquad\qquad y^4\;)\;x^2y^4-y^6\\
\qquad\qquad\qquad x^2-y^2\;)\;x^4-y^4\;(\;x^2+y^2.\\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{x^4-x^2y^2}\\
\qquad\qquad\qquad\qquad\qquad\qquad x^2y^2-y^4\\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^2y^2-y^4}
\end{array}$$

$$x^2-y^2\Big)\frac{x^6-y^6}{x^4-y^4}=\frac{x^4+x^2y^2+y^4}{x^2+y^2}.$$ *Ans.*

Art. 118. (p. 52.)

Ex. (6.) $a+\frac{b^2-cd}{n}=\frac{n\times a+\overline{b^2-cd}}{n}=\frac{an+b^2-cd}{n}.$

(7.) $7x-\frac{4n^2+5a}{8}=\frac{8\times 7x-\overline{4n^2+5a}}{8}=\frac{56x-4n^2-5a}{8}.$

(8.) $15a-\frac{3m^2x-d^2}{4m}=\frac{4m\times 15a-\overline{3m^2x-d^2}}{4m}=\frac{60am-3m^2x+d^2}{4m}.$

(9.) $7a-b-\frac{7e-m}{4n}=\frac{4n\times\overline{7a-b-7e-m}}{4n}=\frac{28an-4bn-7e+m}{4n}.$

(10.) $$11m-4n+\frac{a^2-5n^3}{3m-2n^2}=\frac{\overline{3m-2n^2}\times\overline{11m-4n}+a^2-5n^3}{3m-2n^2}=\frac{33m^2-22mn^2-12mn+3n^3+a^2}{3m-2n^2}.$$

(11.) $$8x^2+5y^2-\frac{d^3+a^2}{2x-3y^2}=\frac{\overline{2x-3y^2}\times\overline{8x^2+5y^2}-\overline{d^3+a^2}}{2x-3y^2}=\frac{16x^3+10xy^2-24x^2y^2-15y^4-d^3-a^2}{2x-3y^2}.$$

Art. 119. (p. 54.)

Ex. (5.) $$\frac{a^3-b^3+x^3}{a+x}=\overline{a^3-b^3+x^3}\div\overline{a+x}=a^2-ax+x^2-\frac{b^3}{a+x}.$$

(6.) $$\frac{x^3+y^3}{x+y}=\overline{x^3+y^3}\div\overline{x+y}=x^2-xy+y^2.$$

(7.) $$\frac{x^3-y^3}{x-y}=\overline{x^3-y^3}\div\overline{x-y}=x^2+xy+y^2.$$

(8.) $$\frac{ax^2-x}{a}=\overline{ax^2-x}\div a=x^2-\frac{x}{a}.$$

Art. 120. (p. 56.)

Ex. (8.) $$\frac{\frac{5}{6}}{7\frac{1}{4}}=\frac{5}{6}\times\frac{4}{29}=\frac{20}{174}=\frac{10}{87}.$$

(9.) $$\frac{a-\frac{x}{2}}{b+\frac{2y}{3}}=\frac{\frac{2a-x}{2}}{\frac{3b+2y}{3}}=\frac{2a-x}{2}\times\frac{3}{3b+2y}=\frac{6a-3x}{6b+4y}.$$

(10.) $$\frac{m-\frac{n}{3}}{x}=\frac{\frac{3m-n}{3}}{\frac{x}{1}}=\frac{3m-n}{3}\times\frac{1}{x}=\frac{3m-n}{3x}.$$

(11.) $$\frac{y-x+\frac{a}{2}}{7\frac{3}{4}}=\frac{\frac{2y-2x+a}{2}}{\frac{31}{4}}=\frac{2y-2x+a}{2}\times\frac{4}{31}=\frac{8y-8x+4a}{62}.$$

Art. 121. (p. 57.)

Ex. (5.) $\frac{4}{9}, \frac{7}{12}, \frac{1}{2}$.

$$\begin{array}{r|l} 3 & 9,\ 12,\ 2 \\ \hline 2 & 3,\ \ 4,\ 2 \\ \hline & 3,\ \ 2,\ 1 \end{array}$$

$3\times2\times3\times2=36$, common denominator.

$$\begin{array}{r|l} & 36 \\ \hline 9 & 4\times4=16\text{, numerator for } \frac{4}{9}=\frac{16}{36}. \\ 12 & 3\times7=21\text{, numerator for } \frac{7}{12}=\frac{21}{36}. \\ 2 & 18\times1=18\text{, numerator for } \frac{1}{2}=\frac{18}{36}. \end{array}$$

Ex. (6.) $\frac{7}{11}, \frac{4}{19}, \frac{5}{7}, \frac{7}{1}$.

$7\times19\times\ 7\times1=\ \ 931$, numerator for $\frac{7}{11}=\frac{931}{1463}$.
$4\times11\times\ 7\times1=\ \ 308$, numerator for $\frac{4}{9}=\frac{308}{1463}$.
$5\times11\times19\times1=\ 1045$, numerator for $\frac{5}{7}=\frac{1045}{1463}$.
$7\times11\times19\times7=10241$, numerator for $\frac{7}{11}=\frac{10241}{1463}$.

$11\times19\times\ 7\times1=1463$, denominator.

Ex. (7.) $\frac{2}{3}$ of $7\frac{1}{4}$, $\frac{2}{11}$ of 5, $=\frac{29}{6}, \frac{10}{11}$.

$29\times11=319=\frac{319}{66}$.
$10\times\ 6=\ 60=\frac{60}{66}$.

$6\times11=\ 66$, common denominator.

Ex. (8.) $\frac{3}{4}$ of $\frac{9}{11}$ of $\frac{17}{1}$, $\frac{1}{2}$ of 9, $=\frac{459}{44}, \frac{9}{2}$.

$459\times\ 2=918=\frac{918}{88}$.
$9\times44=396=\frac{396}{88}$.

$44\times\ 2=88$, denominator.

Ex. (9.) $\dfrac{\frac{3}{8}}{\frac{5}{9}}, \frac{2}{11}$ of $\dfrac{\frac{3}{4}}{7\frac{1}{4}}, =\frac{27}{40}, \frac{6}{319}$.

$27\times319=\ 8613=\frac{8613}{12760}$.
$6\times\ 40=\ 2400=\frac{2400}{12760}$.

$40\times319=12760$, common denominator.

Ex. (10.) $\frac{3x}{y}, \frac{4m}{x}, \frac{a}{b-c}$.

$$3x\times x\times\overline{b-c}=3bx^2-3cx^2=\frac{3bx^2-3cx^2}{bxy-cxy}.$$

$$4m\times y\times\overline{b-c}=4bmy-4cmy=\frac{4bmy-4cmy}{bxy-cxy}.$$

$$a\times y\times x=axy=\frac{axy}{bxy-cxy}.$$

$y\times x\times\overline{b-c}=bxy-cxy$, common denominator.

Ex. (11.) $\frac{a}{x}, \frac{b}{x-2}, \frac{d-3}{y}$.

$$a\times\overline{x-2}\times y=axy-2ay.$$

$$b\times x\times y=bxy.$$

$$\overline{d-3}\times x\times\overline{x-2}=dx^2-3x^2-2dx+6x.$$

$$x\times\overline{x-2}\times y=x^2y-2xy.$$

$$\frac{axy-2ay}{x^2y-2xy}, \frac{bxy}{x^2y-2xy}, \frac{dx^2-3x^2-2dx+6x}{x^2y-2xy}.$$

Ex. (12.) $\frac{a+b}{x}, \frac{3-a}{y-2}, \frac{x-7}{18}$.

$$\overline{a+b}\times\overline{y-2}\times 18=18ay+18by-36a-36b.$$

$$\overline{3-a}\times x\times 18=54x-18ax.$$

$$\overline{x-7}\times x\times\overline{y-2}=x^2y-7xy-2x^2+14x.$$

$$x\times\overline{y-2}\times 18=18xy-36x.$$

$$\frac{18ay+18by-36a-36b}{18xy-36x}, \frac{54x-18ax}{18xy-36x}, \frac{x^2y-7xy-2x^2+14x}{18xy-36x}.$$

Ex. (13.) $\frac{4a}{b-3}, \frac{a}{b}, \frac{d}{x}, \frac{a-b}{x-5}$.

$$\left.\begin{array}{l} 4a\times b\times x\times\overline{x-5}=4abx^2-20abx. \\ a\times\overline{b-3}\times x\times\overline{x-5}=abx^2-3ax^2-5abx+15ax \\ d\times\overline{b-3}\times b\times\overline{x-5}=b^2dx-3bdx-5b^2d+15bd \\ \overline{a-b}\times\overline{b-3}\times b\times x=ab^2x-b^3x-3abx+3b^2x \end{array}\right\}\text{numerators.}$$

$\overline{b-3}\times b\times x\times\overline{x-5}=b^2x^2-3bx^2-5b^2x+15bx$, denominator.

Ex. (14.) $x,\ y,\ \frac{a}{x},\ \frac{a}{y-3}.$

$$\left.\begin{array}{l} x\times1\times x\times\overline{y-3}=x^2y-3x^2 \\ y\times1\times x\times\overline{y-3}=xy^2-3xy \\ a\times1\times1\times\overline{y-3}=ay-3a \\ a\times1\times1\times x \quad =ax \end{array}\right\}\text{numerators.}$$

$$\overline{1\times1\times x\times\overline{y-3}=xy-3x,\ \text{denominator.}}$$

Ex. (15.) $a,\ b,\ c,\ d,\ \frac{a}{b}.$

$$\left.\begin{array}{l} a\times1\times1\times1\times b=ab \\ b\times1\times1\times1\times b=b^2 \\ c\times1\times1\times1\times b=bc \\ d\times1\times1\times1\times b=bd \\ a\times1\times1\times1\times1=a \end{array}\right\}\text{numerators.}$$

$$\overline{1\times1\times1\times1\times b=b,\ \text{denominator.}}$$

Ex. (16.) $\dfrac{\frac{x}{y}}{\frac{m}{2}},\ \dfrac{\frac{x}{3y}}{\frac{1}{m-n}};\quad \dfrac{\frac{x}{y}}{\frac{m}{2}}=\frac{x}{y}\times\frac{2}{m}=\frac{2x}{my}.$

$$\dfrac{\frac{x}{3y}}{\frac{1}{m-n}}=\frac{x}{3y}\times\frac{m-n}{1}=\frac{mx-nx}{3y}.$$

$\frac{2x}{my},\ \frac{mx-nx}{3y}.$ $\quad y\,\underline{)\,my,\ 3y}$
$\qquad\qquad\qquad\qquad m,\ \ 3.$

$y\times m\times3=3my$, denominator.

$$\begin{array}{r|l} & 3my \\ \hline my & 3\times2x \qquad =6x \\ 3y & m\times mx-nx=m^2x-mnx \end{array}\Big\}\ \text{numerators.}$$

Ex. (17.) $\frac{x}{7\frac{1}{2}},\ \frac{5\frac{1}{3}}{x};\quad \frac{x}{7\frac{1}{2}}=\dfrac{\frac{x}{1}}{\frac{15}{2}}=\frac{x}{1}\times\frac{2}{15}=\frac{2x}{15}.$

$$\frac{5\frac{1}{3}}{x}=\frac{\frac{16}{3}}{\frac{x}{1}}=\tfrac{16}{3}\times\frac{1}{x}=\frac{16}{3x}.$$

$$\frac{2x}{15},\ \frac{16}{3x}. \qquad \begin{array}{r} 3\,)\,\underline{15,\ 3x} \\ 5,\ \ x. \end{array}$$

$3\times5\times x=15x$, common denominator.

$$\begin{array}{r|l} & 15x \\ \hline 15 & x\times2x=2x^2 \\ 3x & 5\times16=80 \end{array} \Big\} \text{ numerators.}$$

ART. 122. (p. 60.)

Ex. (4.) $\frac{3a}{5d}, \frac{4m}{7a}, \frac{3e}{4n^2}$.

$$\begin{array}{l} 3a\times7a\times4n^2=84a^2n^2. \\ 4m\times5d\times4n^2=80dmn^2. \\ 3e\times5d\times7a=105ade. \\ \hline 5d\times7a\times4n^2=\dfrac{84a^2n^2+80dmn^2+105ade}{140adn^2}. \end{array}$$

Ex. (5.) $\frac{7}{8}, \frac{5}{12}, \frac{4}{9}$.

$$\begin{array}{r} 4\,)\,\underline{8,\ 12,\ 9} \\ 3\,)\,\underline{2,\ \ 3,\ 9} \\ 2,\ \ 1,\ 3 \end{array} \qquad \begin{array}{r|l} \multicolumn{2}{r}{4\times3\times2\times3=72} \\ & 72 \\ \hline 8 & 9\times7=63 \\ 12 & 6\times5=30 \\ 9 & 8\times4=32 \\ \hline \multicolumn{2}{l}{\dfrac{125}{72}=1\tfrac{53}{72}.} \end{array}$$

Ex. (6.) $\frac{5}{8}, \frac{7}{11}, \frac{4}{5}$.

$$\begin{array}{l} 5\times11\times\ 5=275 \\ 7\times\ 8\times\ 5=280 \\ 4\times\ 8\times11=352 \\ \hline 8\times11\times\ 5=\dfrac{907}{440}=2\tfrac{27}{440}. \end{array}$$

Ex. (7.) $\frac{8}{9}, \frac{2}{3}, \frac{5}{6}, \frac{7}{12}$.

$$\begin{array}{r|l} 3 & 9,\ 3,\ 6,\ 12 \\ \hline 2 & 3,\ 1,\ 2,\ \ 4 \\ \hline & 3,\ 1,\ 1,\ \ 2 \end{array}$$

$3\times2\times3\times2=36$, common denominator.

$$\begin{array}{r|l} & 36 \\ \hline 9 & 4\times8=32 \\ 3 & 12\times2=24 \\ 6 & 6\times5=30 \\ 12 & 3\times7=21 \\ \hline & \end{array}$$

$$\frac{107}{36}=2\tfrac{35}{36}.\quad \textit{Ans.}$$

Ex. (8.) $8\frac{3}{4}, 3\frac{2}{3}, 7\frac{5}{6}=\frac{35}{4}, \frac{11}{3}, \frac{47}{6}$

$$\begin{array}{r} 35\times3\times6=630 \\ 11\times4\times6=264 \\ 47\times4\times3=564 \\ \hline \end{array}$$

$$4\times3\times6=\ \frac{1458}{72}=20\tfrac{1}{4}.\quad \textit{Ans.}$$

Ex. (9.)

$\frac{2}{3}$ of $7\frac{1}{4}$, $\frac{7}{11}$ of 13, $=\frac{29}{6}, \frac{91}{11}$.

$$\begin{array}{r} 29\times11=319 \\ 91\times\ 6=546 \\ \hline \end{array}$$

$$6\times11=\ \frac{865}{66}=13\tfrac{7}{66}.\quad \textit{Ans.}$$

Ex. (10.)

$\frac{2}{5}$ of 1, $\frac{1}{2}$ of $\frac{2}{3}$, $=\frac{2}{5}, \frac{1}{3}$.

$$\begin{array}{r} 2\times3=\ 6 \\ 1\times5=\ 5 \\ \hline \end{array}$$

$$5\times3=\ \frac{11}{15}.\quad \textit{Ans.}$$

Ex. (11.) $\dfrac{\frac{3}{4}}{\frac{7}{1}}, \dfrac{\frac{1}{2}}{\frac{7}{11}}=\dfrac{\frac{3}{4}}{\frac{7}{1}}=\frac{3}{4}\times\frac{1}{7}=\frac{3}{28}, \dfrac{\frac{1}{2}}{\frac{7}{11}}=\frac{1}{2}\times\frac{11}{7}=\frac{11}{14}, \frac{3}{28}, \frac{11}{14}$.

$$\begin{array}{r} 3\times14=\ 42 \\ 11\times28=308 \\ \hline \end{array}$$

$$28\times14\ =\frac{350}{392}=\frac{25}{28}.\quad \textit{Ans.}$$

Ex. (12.) $\frac{3}{4}$ of $\frac{4\frac{2}{3}}{11\frac{1}{2}}$, $\frac{2}{9}$ of $\frac{\frac{3}{4}}{7\frac{2}{5}}$.

$$\tfrac{3}{4}\times\frac{4\frac{2}{3}}{11\frac{1}{2}}=\tfrac{3}{4}\times\frac{\frac{14}{3}}{\frac{23}{2}}=\tfrac{3}{4}\times\tfrac{14}{3}\times\tfrac{2}{23}=\tfrac{84}{276}=\tfrac{7}{23}.$$

$$\tfrac{2}{9}\times\frac{\frac{3}{4}}{7\frac{2}{5}}=\tfrac{2}{9}\times\frac{\frac{3}{4}}{\frac{37}{5}}=\tfrac{2}{9}\times\tfrac{3}{4}\times\tfrac{5}{37}=\tfrac{30}{1332}=\tfrac{5}{222}.$$

$$\tfrac{7}{23},\ \tfrac{5}{222}.$$

$$\begin{array}{r}
7\times222=1554\\
5\times\ 23=\ 115\\
\hline
1669\\
23\times222=\overline{5106}.
\end{array}$$

Ex. (13.) $\frac{3x}{4a}$, $\frac{2x}{3e}$.

$$\begin{array}{r}
3x\times3e=9ex\\
2x\times4a=8ax\\
\hline
9ex+8ax\\
4a\times3e=\overline{12ae}.
\end{array}\quad \textit{Ans.}$$

Ex. (14.) $\frac{x}{3}$, $\frac{x}{4}$, $\frac{x}{5}$.

$$\begin{array}{r}
x\times4\times5=20x\\
x\times3\times5=15x\\
x\times3\times4=12x\\
\hline
47x\\
3\times4\times5=\overline{60}.
\end{array}\quad \textit{Ans.}$$

Ex. (15.) $\frac{4a}{7}$, $\frac{a-3}{4}$.

$$\begin{array}{r}
4a\times4=16a\\
\overline{a-3}\times7=\ 7a-21\\
\hline
23a-21\\
7\times4=\overline{28}.
\end{array}\quad \textit{Ans.}$$

Ex. (16.) $4m$, $\frac{3a-1}{2}$, $\frac{4n+2}{3}$.

$$\begin{array}{r}
4m\times2\times3=24m\\
\overline{3a-1}\times1\times3=\ 9a-3\\
\overline{4n+2}\times1\times2=\ 8n+4\\
\hline
9a+24m+8n+1\\
1\times2\times3=\overline{\qquad\qquad6\qquad\qquad}.
\end{array}\quad \textit{Ans.}$$

Ex. (17.) $\frac{4a-3}{5}, \frac{7a+1}{3}, \frac{3a}{2}.$

$$\overline{4a-3}\times3\times2=24a-18$$
$$\overline{7a+1}\times5\times2=70a+10$$
$$3a\quad\times5\times3=45a$$
$$5\times3\times2=\frac{139a-8}{30}.\quad \textit{Ans.}$$

Ex. (18.) $\frac{3}{a+b}, \frac{3}{a-b}.$

$$3\times\overline{a-b}=3a-3b$$
$$3\times\overline{a+b}=3a+3b$$
$$\overline{a+b}\times\overline{a-b}=\frac{6a}{a^2-b^2}.\quad \textit{Ans}$$

Ex. (19.) $\frac{a}{a-b}, \frac{a-b}{c+d}, \frac{a+b}{c-d}.$

$$a\quad\times\overline{c+d}\times\overline{c-d}=ac^2-ad^2,$$
$$\overline{a-b}\times\overline{a-b}\times\overline{c-d}=a^2c-2abc+b^2c-a^2d+2abd-b^2d$$
$$\overline{a+b}\times\overline{a-b}\times\overline{c+d}=a^2c-b^2c+a^2d-b^2d$$
$$\overline{a-b}\times\overline{c+d}\times\overline{c-d}=\frac{2a^2c+ac^2-ad^2-2abc+2abd-2b^2d}{ac^2-ad^2-bc^2+bd^2}.\quad \textit{Ans.}$$

Ex. (20.) $\frac{\frac{1}{3b^2}}{\frac{a-b}{2}}, \frac{\frac{b}{c}}{\frac{2a-b}{3}};\quad \frac{\frac{1}{3b^2}}{\frac{a-b}{2}}=\frac{1}{3b^2}\times\frac{2}{a-b}=\frac{2}{3ab^2-3b^3}.$

$$\frac{\frac{b}{c}}{\frac{2a-b}{3}}=\frac{b}{c}\times\frac{3}{2a-b}=\frac{3b}{2ac-bc}.$$

$$\frac{2}{3ab^2-3b^3}, \frac{3b}{2ac-bc}.$$

$$2\times\overline{2ac-bc}=4ac-2bc$$
$$3b\times\overline{3ab^2-3b^3}=9ab^3-9b^4$$
$$\overline{3ab^2-3b^3}\times\overline{2ac-bc}=\frac{4ac-2bc+9ab^3-9b^4}{6a^2b^2c-9ab^3c+3b^4c}.\quad \textit{Ans.}$$

ART. 123. (p. 62.)

Ex. (3.) $\frac{7}{8}-\frac{4}{11}$.

$7\times11=77$

$4\times\ 8=32$

$8\times11=\ \frac{45}{88}$. *Ans.*

Ex. (4.) $\frac{2}{3}-\frac{4}{9}$.

$2\times9=18$

$4\times3=12$

$3\times9=\ \frac{6}{27}=\frac{2}{9}$. *Ans.*

Ex. (5.) $7\frac{3}{4}-4\frac{1}{4}=\frac{31}{4}-\frac{17}{4}=\frac{14}{4}=3\frac{1}{2}$. *Ans.*

Ex. (6.)

$6\frac{3}{11}-\frac{2}{3}$ of $5=\frac{69}{11}-\frac{10}{3}$.

$69\times\ 3=207$

$10\times11=110$

$11\times3=\ \frac{97}{33}=2\frac{31}{33}$. *Ans.*

Ex. (7.) $8\frac{1}{7}-\frac{2}{5}$ of $17\frac{1}{2}=\frac{57}{7}-\frac{35}{5}=\frac{57}{7}-\frac{7}{1}$.

$\frac{57}{7}-\frac{7}{1}$.

$57\times1=57$

$7\times7=49$

$7\times1=\ \frac{8}{7}=1\frac{1}{7}$.

Ex. (8.) $\frac{3}{7}$ of $11\frac{2}{15}-\frac{11}{12}$ of $3\frac{1}{2}$.

$\frac{3}{7}$ of $11\frac{2}{15}=\frac{167}{35}$.

$\frac{11}{12}$ of $3\frac{1}{2}=\frac{77}{24}$.

$\frac{167}{35}-\frac{77}{24}$.

$167\times24=4008$

$77\times35=2695$

$35\times24=\ \frac{1313}{840}=1\frac{473}{840}$.

Ex. (9.)

$\frac{7}{8}$ of $13\frac{3}{7}-\frac{4}{11}$ of $7\frac{1}{8}=\frac{329}{28}-\frac{57}{22}$.

$329\times22=7238$

$57\times28=1596$

$28\times22=\ \frac{5642}{616}=9\frac{7}{44}$. *Ans.*

Ex. (10.)

$\frac{3}{5}$ of $7-\frac{1}{3}$ of $17\frac{1}{4}=\frac{21}{5}-\frac{23}{12}$.

$21\times12=252$

$23\times\ 5=115$

$5\times12=\ \frac{137}{60}=2\frac{17}{60}$. *Ans*

Ex. (11.) $\dfrac{7\frac{2}{3}}{9\frac{1}{4}}-\dfrac{\frac{2}{3}}{4\frac{2}{5}}=\dfrac{\frac{23}{3}}{\frac{37}{4}}-\dfrac{\frac{2}{3}}{\frac{22}{5}}=\overline{\frac{23}{3}\times\frac{4}{37}}-\overline{\frac{2}{3}\times\frac{5}{22}}=\frac{92}{111}-\frac{5}{33}$.

$92\times\ 33=3036$

$5\times111=\ 555$

$111\times33=\ \frac{2481}{3663}=\frac{827}{1221}$. *Ans.*

Ex. (12.) $\frac{2}{a-1}-\frac{2}{a+1}$.

$$2\times\overline{a+1}=2a+2$$
$$2\times\overline{a-1}=2a-2$$
$$\overline{a-1}\times\overline{a+1}=\frac{4}{a^2-1}.$$ *Ans.*

Ex. (13.) $\frac{7x}{5}-\frac{4x}{7}$.

$$7x\times7=49x$$
$$4x\times5=20x$$
$$5\times7=\frac{29x}{35}.$$ *Ans.*

Ex. (14.) $\frac{3a-2b}{3c}-\frac{2a-4b}{5b}$.

$$\overline{3a-2b}\times5b=15ab-10b^2$$
$$\overline{2a-4b}\times3c=6ac-12bc$$
$$3c\times5b=\frac{15ab-10b^2-6ac+12bc}{15bc}.$$ *Ans.*

Ex. (15.) $\frac{12x}{5}-\frac{3x}{7}$.

$$12x\times7=84x$$
$$3x\times5=15x$$
$$5\times7=\frac{69x}{35}.$$ *Ans.*

Ex. (16.) $\frac{x+y}{x-y}-\frac{x-y}{x+y}$.

$$\overline{x+y}\times\overline{x+y}=x^2+2xy+y^2$$
$$\overline{x-y}\times\overline{x-y}=x^2-2xy+y^2$$
$$\overline{x-y}\times\overline{x+y}=\frac{4xy}{x^2-y^2}.$$ *Ans.*

Ex. (17.)

$$3a+\frac{a+b}{d}-\left(a-\frac{a-b}{d}\right)=$$
$$2a+\frac{2a}{d}.$$ *Ans.*

Ex. (18.)

$$7x-\frac{3a-2b}{3}-\left(x-\frac{4a-b}{2}\right)=$$
$$\frac{21x-3a+2b}{3}-\frac{2x-4a+b}{2}.$$
$$\overline{21x-3a+2b}\times2=42x-6a+4b$$
$$\overline{2x-4a+b}\times3=6x-12a+3b$$
$$3\times2=\frac{36x+6a+b}{6}=$$
$$6x+a+\frac{b}{6}.$$ *Ans.*

Ex. (19.)

$$\frac{(a+b)^2}{ab}-\frac{(a-b)^2}{ab}=$$
$$\frac{a^2+2ab+b^2}{ab}-\frac{a^2-2ab+b^2}{ab}$$
$$=4.$$ *Ans.*

Ex. (20.) $$\frac{\frac{a+b}{2}}{\frac{a-b}{3}}-\frac{\frac{a-b}{3}}{\frac{a+b}{2}}=\frac{a+b}{2}\times\frac{3}{a-b}-\frac{a-b}{3}\times\frac{2}{a+b}=$$

$$\frac{3a+3b}{2a-2b}-\frac{2a-2b}{3a+3b}.$$

$$\begin{array}{r}\overline{3a+3b}\times\overline{3a+3b}=9a^2+18ab+9b^2\\ \overline{2a-2b}\times\overline{2a-2b}=4a^2-\ \ 8ab+4b^2\\ \hline \overline{2a-2b}\times\overline{3a+3b}=\dfrac{5a^2+26ab+5b^2}{6a^2-6b^2}.\end{array}$$

ART. 124. (p. 64.)

If the indices are fractions, add them.

Ex. (17.) $$\frac{ab^{\frac{3}{4}}}{mn^{\frac{1}{2}}}\times\frac{b^{\frac{1}{2}}}{mn^{\frac{1}{3}}}=\frac{ab^{\frac{5}{4}}}{m^2n^{\frac{5}{6}}}.$$

$$\tfrac{3}{4}+\tfrac{1}{2}=\tfrac{5}{4};\ \tfrac{1}{2}+\tfrac{1}{3}=\tfrac{5}{6}.$$

Ex. (18.) $\frac{mn^a}{4hy}\times\frac{hy}{mn^a}=\frac{1}{4}$. *Ans.* | Ex. (19.) $\frac{a^n}{b}\times\frac{a^m}{h}=\frac{a^{m+n}}{bh}$. *Ans.*

ART. 125. (p. 66.)

Ex. (11.) $$\frac{4x}{2x-1}\div\frac{2}{x+1}=\frac{4x}{2x-1}\times\frac{x+1}{2}=\frac{4x^2+4x}{4x-2}=\frac{2x^2+2x}{2x-1}.$$

(12.) $$\frac{2a-b}{4ac}\div\frac{2ac}{3b-1}=\frac{2a-b}{4ac}\times\frac{3b-1}{2ac}=\frac{6ab+b-3b^2-2a}{8a^2c^2}.\ \textit{Ans.}$$

(13.) $$\frac{5a^4-5b^4}{2a^2-4ab+2b^2}\div\frac{6a^2+5ab}{4a-4b}=\frac{5a^4-5b^4}{2a^2-4ab+2b^2}\times\frac{4a-4b}{6a^2+5ab}=$$

$$\frac{20a^5-20ab^4-20a^4b+20b^5}{12a^4-14a^3b-8a^2b^2+10ab^3}.\ \textit{Ans.}$$

SIMPLE EQUATIONS.

ART. 142. (p. 72.)

Ex. (9.) Given $4x-5=71+8$ to find x.

Conditions, $4x-5=71+8.$
Transposing, $4x=71+8+5.$
Uniting, $4x=84.$
Dividing, $x=21.$

Ex. (10.) Given $6x-17-7=0$ to find x.

Conditions, $6x-17-7=0.$
Transposing, $6x=17+7=24.$
Dividing, $x=4.$

Ex. (11.) Given $5x+28+8=6$ to find the value of x.

Conditions, $5x+28+8=6.$
Transposing, $5x=6-28-8.$
Uniting, $5x=-30.$
Dividing, $x=-6.$

Ex. (12.) Given $7x-17+3=100$ to find the value of x.

Conditions, $7x-17+3=100.$
Transposing, $7x=100+17-3.$
Uniting, $7x=114.$
Dividing, $x=16\frac{2}{7}.$

Ex. (13.) Given $23x-96+1=0$ to find the value of x.

Conditions, $23x-96+1=0.$
Transposing, $23x=96-1.$
Uniting, $23x=95.$
Dividing, $x=4\frac{3}{23}.$

Ex. (14.) Given $17x-7-5-8=4$ to find the value of x.

Conditions, $17x-7-5-8=4.$
Transposing, $17x=4+7+5+8.$
Uniting, $17x=24.$
Dividing, $x=1\frac{7}{17}.$

Ex. (15.) Given $9x=7+8+10$ to find the value of x.

Conditions, $9x=7+8+10.$
Uniting, $9x=25.$
Dividing, $x=2\frac{7}{9}.$

Ex. (16.) Given $7x-10=5x+14$ to find the value of x.

Conditions, $7x-10=5x+14.$
Transposing, $7x-5x=14+10.$
Uniting, $2x=24.$
Dividing, $x=12.$

ART. 148. (p. 76.)

EXAMPLES IN SIMPLE EQUATIONS.

Ex. (1.) Given $5x+22-2x=31$ to find the value of x.

Conditions, $5x+22-2x=31.$
Transposing, $5x-2x=31-22.$
Uniting, $3x=9.$
Dividing, $x=3.$

Ex. (2.) Given $4-19x=14-21x$ to find the value of x.

Conditions, $4-19x=14-21x.$
Transposing, $21x-19x=14-4.$
Uniting, $2x=10.$
Dividing, $x=5.$

Ex. (3.) Given $24x-12=240-12x$ to find the value of x.

Conditions, $24x-12=240-12x.$
Transposing, $24x+12x=240+12.$
Uniting, $36x=252.$
Dividing, $x=7.$

Ex. (4.) Given $15x+7x-10=12x+90$ to find the value of x.

Conditions, $15x+7x-10=12x+90.$
Transposing, $15x+7x-12x=90+10.$
Uniting, $10x=100.$
Dividing, $x=10.$

Ex. (5.) Given $7x+2x=12x-36$ to find x.

Conditions, $7x+2x=12x-36.$
Transposing and uniting, $-3x=-36.$
Dividing by -3, $x=12.$

Ex. (6.) Given $12x-3x-2x=63$ to find x.

Conditions, $12x-3x-2x=63.$
Uniting terms, $7x=63.$
Dividing, $x=9.$

Ex. (7.) Given $x+\frac{x}{4}+\frac{x}{5}=87$ to find x.

Conditions, $x+\frac{x}{4}+\frac{x}{5}=87.$
Clearing of fractions, $20x+5x+4x=1740.$
Uniting, $29x=1740.$
Dividing, $x=60.$

Ex. (8.) Given $x-\frac{x}{4}+13=\frac{x}{2}+40$ to find x.

Conditions, $x-\frac{x}{4}+13=\frac{x}{2}+40.$
Clearing of fractions, $4x-x+52=2x+160.$
Transposing, $4x-x-2x=160-52.$
Uniting, $x=108.$

Ex. (9.) Given $\frac{x}{5}+\frac{x}{12}=\frac{x}{10}+22$ to find x.

Conditions, $\frac{x}{5}+\frac{x}{12}=\frac{x}{10}+22.$
Clearing of fractions, $12x+5x=6x+1320.$
Transposing, $12x+5x-6x=1320.$
Uniting, $11x=1320.$
Dividing, $x=120.$

Ex. (10.) Given $x-\frac{x}{7}+20=\frac{x}{2}+\frac{x}{4}+26$ to find x.

Conditions, $x-\frac{x}{7}+20=\frac{x}{2}+\frac{x}{4}+26.$

Clearing of fractions, $28x-4x+560=14x+7x+728.$

Transposing, $28x-4x-14x-7x=728-560.$

Uniting, $3x=168.$

Dividing, $x=56.$

Ex. (11.) Given $3x+\frac{3x}{4}+15=\frac{x}{2}+41$ to find x.

Conditions, $3x+\frac{3x}{4}+15=\frac{x}{2}+41.$

Clearing of fractions, $12x+3x+60=2x+164.$

Transposing, $12x+3x-2x=164-60.$

Uniting, $13x=104.$

Dividing, $x=8.$

Ex. (12.) Given $x-\frac{4x+8}{6}=8$ to find x.

Conditions, $x-\frac{4x+8}{6}=8.$

Clearing of fractions, $6x-4x-8=48.$

Uniting, $2x=56.$

Dividing, $x=28.$

Ex. (13.) Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$ to find x.

Conditions, $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}.$

Clearing of fractions, $336+3x-11=10x-10+776-56x$

Transposing, $3x+56x-10x=776-10+11-336.$

Uniting, $49x=441.$

Dividing, $x=9.$

Ex. (14.) Given $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$ to find x.

Conditions, $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}.$

Clearing of fractions, $6x+9x-15=72-4x+8.$

Transposing, $6x+9x+4x=72+15+8.$

Uniting, $19x=95.$

Dividing, $x=5.$

Ex. (15.) Given $17x-\frac{5x-4}{3}-\frac{8x+4}{5}=20x-\frac{3x+8}{2}-5$ to find the value of x.

Conditions, $17x-\frac{5x-4}{3}-\frac{8x+4}{5}=20x-\frac{3x+8}{2}-5.$

Clearing of fractions,

$$510x-50x+40-48x-24=600x-45x-120-150.$$

Transposing,

$$510x+45x-50x-48x-600x=24-40-120-150.$$

Uniting, $-143x=-286.$

Dividing by -143, $x=2.$

Ex. (16.) Given $9x-\frac{x-1}{2}+\frac{2x-2}{3}=12x-\frac{5x-7}{4}-13$ to find x.

Conditions, $9x-\frac{x-1}{2}+\frac{2x-2}{3}=12x-\frac{5x-7}{4}-13.$

Clearing of fractions,

$$108x-6x+6+8x-8=144x-15x+21-156.$$

Uniting, &c., $-19x=-133.$

Dividing by -19, $x=7.$

Ex. (17.) Given $x+\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=2x+17$ to find x.

Conditions, $x+\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=2x+17.$

Clearing of fractions and uniting, $137x=120x+1020.$

Transposing, $137x-120x=1020.$

Uniting, $17x=1020.$

Dividing, $x=60.$

Ex. (18.) Given $\frac{a}{x}=b+c$ to find x.

Conditions, $\frac{a}{x}=b+c.$

Clearing of fractions, $a=bx+cx.$

Transposing, $bx+cx=a.$

Dividing, $x=\frac{a}{b+c}.$

Ex. (19.) Given $8x-40=0$ to find x.

Conditions, $8x-40=0.$

Transposing, $8x=40.$

Dividing, $x=5.$

Ex. (20.) Given $a+\frac{1}{x}=b+c+\frac{d}{x}$ to find x.

Conditions, $a+\frac{1}{x}=b+c+\frac{d}{x}.$

Clearing of fractions, $ax+1=bx+cx+d.$

Transposing, $ax-bx-cx=d-1.$

Dividing, $x=\frac{d-1}{a-b-c}.$

Ex. (21.) Given $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$ to find x.

Conditions, $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}.$

Clearing of fractions,

$$70x-42x+42+280=700-35x-60x+80+56x-56.$$

Transposing, &c., $165x-98x=780-378.$

Uniting, $67x=402.$

Dividing, $x=6.$

Ex. (22.) Given $ax^2+bx=mx^2+nx$ to find x.

Conditions, $ax^2+bx=mx^2+nx.$

Dividing by x, $ax+b=mx+n.$

Transposing, $ax-mx=n-b.$

Dividing, $x=\frac{n-b}{a-m}.$

Ex. (23.) Given $ax+m=bx+n$ to find the value of x.

Conditions, $ax+m=bx+n.$

Transposing, $ax-bx=n-m.$

Dividing, $x=\frac{n-m}{a-b}.$

Ex. (24.) Given $\frac{3x}{b}-\frac{x}{c}=m-c$ to find the value of x.

Conditions, $\frac{3x}{b}-\frac{x}{c}=m-c.$

Clearing of fractions, $3cx-bx=bcm-bc^2.$

Dividing, $x=\frac{bcm-bc^2}{3c-b}=\frac{bc(m-c)}{3c-b}.$

Ex. (25.) Given $\frac{7x}{a}-\frac{3}{m}=15x+n$ to find the value of x.

Conditions, $\frac{7x}{a}-\frac{3}{m}=15x+n.$

Clearing of fractions, $7mx-3a=15amx+amn.$

Transposing, $7mx-15amx=amn+3a.$

Dividing, $x=\frac{amn+3a}{7m-15am}.$

Ex. (26.) Given $\frac{a-x}{b}-\frac{4a-x}{c}=a-b$ to find the value of x.

Conditions, $\frac{a-x}{b}-\frac{4a-x}{c}=a-b.$

Clearing of fractions, $ac-cx-4ab+bx=abc-b^2c.$

Transposing, $bx-cx=-ac+4ab+abc-b^2c.$

Dividing, $x=\frac{4ab-ac+abc-b^2c}{b-c}.$

Ex. (27.) Given $\frac{a^2x}{b-c}+de=3x-\frac{d}{e}$ to find the value of x.

Conditions, $$\frac{a^2x}{b-c}+de=3x-\frac{d}{e}.$$

Clearing of fractions, $$a^2ex+bde^2-cde^2=3bex-3cex-bd+cd.$$

Transposing, $$a^2ex-3bex+3cex=cde^2-bde^2-bd+cd.$$

Dividing, $$x=\frac{cde^2-bde^2-bd+cd}{a^2e+3ce-3be}.$$

Ex. (28.) Given $5x-\frac{4x-a}{b}+\frac{2x+2a}{4}=m+n-\frac{2x+2a}{c}$ to find the value of x.

Conditions, $$5x-\frac{4x-a}{b}+\frac{2x+2a}{4}=m+n-\frac{2x+2a}{c}.$$

Clearing of fractions,

$$20bcx-16cx+4ac+2bcx+2abc=4bcm+4bcn-8bx-8ab.$$

Transposing, $$22bcx-16cx+8bx=4bcm+4bcn-8ab-4ac-2abc.$$

Dividing, $$x=\frac{4bcm+4bcn-8ab-4ac-2abc}{22bc-16c+8b}.$$

Reducing, $$x=\frac{2bcm+2bcn-4ab-2ac-abc}{11bc-8c+4b}.$$

Ex. (29.) Given $\frac{6x+18}{13}-4\frac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}-\frac{21-2x}{18}$ to find the value of x.

Conditions, $$\frac{6x+18}{13}-4\tfrac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}-\frac{21-2x}{18}.$$

Multiply by 36,

$$\frac{216x+648}{13}-174-11+3x=180x-1728-39+3x-42+4x.$$

Multiply by 13, $$216x+648-2262-143+39x=2340x-22464$$
$$-507+39x-546+52x.$$

Transposing and reducing, $$255x-2431x=2405-24165.$$

Dividing by -2176, $$-2176x=-21760.$$

$$x=10.$$

Ex. (30.) Given $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$ to find the value of x.

Conditions, $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$.

Multiplying by 18, $8x+6+\frac{126x-522}{5x-12}=8x+19$.

Transposing and reducing, $\frac{126x-522}{5x-12}=13$.

Multiplying by $5x-12$, $126x-522=65x-156$.

Transposing and reducing, $x=6$.

PROBLEMS IN SIMPLE EQUATIONS.

ART. 148. (p. 82.)

Ex. (13.) Let x = first carriage; $5x$ = second carriage; then, $6x$ = the horse.

Therefore, $x+5x+6x=12x=300$.

Dividing, $x=25$, value of the first carriage.
Multiplying, $5x=125$, value of the second carriage
" $6x=150$, value of the horse.

Ex. (14.) Let x = the age of the daughter.
Then, $3x$ = the age of the wife.
And, $6x$ = the age of the gentleman.

Therefore, $10x=120$.
Dividing, $x=12$, daughter's age.
Multiplying, $3x=36$, wife's age.
Multiplying, $6x=72$, gentleman's age.

Ex. (15.) Let x = the sum given the second beggar.
Then, $2x$ = the sum given the first beggar.
Then, $3x$ = the sum given the third beggar.
And, $5x$ = the sum given the fourth beggar.

Then, $11x=77$ cents.
Dividing, $x=7$, given the second.
Multiplying, $2x=14$, given the first.
" $3x=21$, given the third.
" $5x=35$, given the fourth.

Ex. (16.) Let x = oxen, and $2x$ = cows.

Then, $\overline{55\times x}+\overline{32\times 2x}=\$1428.$
And $55x+64x=\$1428.$
Reducing, $119x=\$1428.$
Dividing, $x=12$ oxen.
Multiplying, $2x=24$ cows.

Ex. (17.) Let x = a son's portion; then $2x$ = a daughter's; and $6x$ = the wife's.

Therefore, $\overline{3\times x}+\overline{2\times 2x}+6x=\$1872.$
And $3x+4x+6x=\$1872.$
Collecting, $13x=\$1872.$
Dividing, $x=144$, a son's portion.
Multiplying, $2x=288$, a daughter's.
$6x=864$, the wife's.

Ex. (18.) Let x = apples; then $2x$ = oranges; $6x$ = pears.

Therefore, $\overline{2\times x}+\overline{3\times 2x}+\overline{4\times 6x}=\$2.24.$
Reducing, $32x=224.$
Dividing, $x=7$, the apples.
Multiplying, $2x=14$, the oranges.
Multiplying, $6x=42$, the pears.

Ex. (19.) Let x = the less part, and $4x$ = the larger.

Then, $x+4x=85.$
Uniting, $5x=85.$
Dividing, $x=17$, the less part.
Multiplying, $4x=68$, the larger part.

Ex. (20.) Let x = C's share; $x+10$ = B's; and $x+x+10$ = A's.

Then, $x+(x+10)+(x+x+10)=100.$
Reducing, $4x+20=100.$
Uniting, $4x=80.$
Dividing, $x=20$, C's share.
$20+10=30$, B's share.
$30+20=50$, A's share.

Ex. (21.) Let x = A's share, and $1000-x$ = B's.

Then, $x : 1000-x :: 7 : 8$.

Multiplying extremes, &c., $8x=7000-7x$.

Transposing, $15x=7000$.

Dividing, $x=466\frac{2}{3}$, A's share.

$1000-466\frac{2}{3}=533\frac{1}{3}$, B's share.

Ex. (22.) Let x = the number.

Then, $\frac{x}{3}-6\frac{2}{5}=\frac{x}{5}$.

Clearing of fractions, $5x-96=3x$.

Transposing, &c., $2x=96$.

Dividing, $x=48$.

Ex. (23.) Let x = the days he labored; $36-x$ = the days he was absent.

Then, $125\times x-50(36-x)=1700$.

Reducing, $125x-1800+50x=1700$.

Collecting and transposing, $175x=3500$.

Dividing, $x=20$, the days he labored.

$36-20=16$, the days he was absent.

Ex. (24.) Let x = the gallons the cask contained.

Then, $x-\frac{x}{3}-13=\frac{x}{2}$.

Clearing of fractions, $6x-2x-78=3x$.

Transposing, &c., $x=78$ gallons.

Ex. (25.) Let x = the larger part; $30-x$ = the less.

Then, $\frac{2x}{3}-6\frac{2}{3}=\frac{4}{9}(30-x)$.

Clearing of fractions, $6x-60=120-4x$.

Transposing and uniting, $10x=180$.

Dividing, $x=18$, the larger.

$30-18=12$, the less.

Ex. (26.) Let x = the larger, and $x-3$ = the less.

Then, $x^2-(x-3)^2=51$.
Involving $(x-3)$, $x^2-(x^2-6x+9)=51$.
Transposing and uniting, $6x=60$.
Dividing, $x=10$, the larger.
$10-3=7$, the less.

Ex. (27.) Let x = the sum A put in; then $2x$ = the sum B put in; and $3(x+2x)$ = the sum C put in.

Then, $x+2x+3(x+2x)=864$.
Collecting terms, $12x=864$.
Dividing, $x=72$, A put in.
Multiplying, $2x=144$, B put in.
Multiplying, $9x=648$, C put in.

Ex. (28.) Let x = James' apples, and $(44-x)$ William's.

Then, $\frac{2}{9}(x+12)=44-x-12$.
Multiplying, $\frac{2x+24}{9}=32-x$.
Clearing of fractions, $2x+24=288-9x$.
Uniting terms, &c., $11x=264$.
Dividing, $x=24$, James' apples.
$44-24=20$, William's.

Or,

$x-12=\frac{3}{8}(44-x+12)$.
Clearing of fractions, &c., $8x-96=132-3x+36$.
Uniting, &c., $11x=264$.
Dividing, $x=24$, James'.
$44-24=20$, William's.

Ex. (29.) Let x = the larger, and $112-x$ = the less.

Then, $9 : 7 :: x : (112-x)$.
Multiplying extremes, &c., $9(112-x)=7x$.
Multiplying, $1008-9x=7x$.
Transposing, &c., $16x=1008$.
Dividing, $x=63$, the larger.
$112-63=49$, the less.

Ex. (30.) Let x = the greater part, and $19-x$ = the less.
Then, $3x=4(19-x)$.
Multiplying, $3x=76-4x$.
Transposing, &c., $7x=76$.
Dividing, $x=10\frac{6}{7}$, the greater.
$19-10\frac{6}{7}=8\frac{1}{7}$, the less.

Ex. (31.) Let x = the larger, and $24-x$ the less.
Then, $x+7 : (24-x)+4 :: 4 : 3$.
Multiplying extremes, &c., $3x+21=112-4x$.
Transposing and uniting, $7x=91$.
Dividing, $x=13$, the larger.
$24-13=11$, the less.

Ex. (32.) Let x = the larger, and $(x-4)$ = the less
Then, $7x=11(x-4)$.
Multiplying, $7x=11x-44$.
Changing terms, &c., $4x=44$.
Dividing, $x=11$, the larger.
$11-4=7$, the less.

Ex. (33.) Let x = the bushels of the first kind, and $(80-x)$ = the bushels of the second kind.
Then, $\frac{250x+200(80-x)}{80}=210$.
Clearing of fractions, $250x+16000-200x=16800$.
Uniting terms, $50x=800$.
Dividing, $x=16$ bushels, 1st kind.
$80-16=64$ bushels, 2nd kind.

Ex. (34.) Let x = his money.
Then, $x-\frac{x}{4}=96$.
Clearing of fractions, $4x-x=384$.
Uniting terms, $3x=384$.
Dividing, $x=128$, the sum at first.

Ex. (35.) Let x = the sum he had at first.

Then, $$x+\frac{x}{2}-40=\frac{4}{5}\times\frac{x}{2}=\frac{4x}{10}.$$

Clearing of fractions, $$10x+5x-400=4x.$$

Uniting and transposing, $$11x=400.$$

Dividing; $$x=36\tfrac{4}{11}.$$

Ex. (36.) Let x = the pupils in the school; then $\frac{2x}{5}$ study grammar; $x-\frac{2x}{5}=\frac{3x}{5}$; $\frac{2}{3}\times\frac{3x}{5}=\frac{6x}{15}=\frac{2x}{5}$ read; 10 spell; and $\frac{1}{7}\times\frac{2x}{5}=\frac{2x}{35}$ study navigation.

Therefore, $$\frac{2x}{5}+\frac{2x}{5}+10+\frac{2x}{35}=x.$$

Clearing of fractions, $$14x+14x+350+2x=35x.$$

Uniting and changing terms, $$x=70.$$

Ex. (37.) Let x = the sum lent; then $x+\frac{x}{20}=\frac{21x}{20}$, amount at the end of the first year. $\frac{21x}{20}+\frac{1}{20}\times\frac{21x}{20}=\frac{21x}{20}+\frac{21x}{400}=\frac{441x}{400}$, amount at the end of the second year. $\frac{441x}{400}+\frac{1}{20}\times\frac{441x}{400}=\frac{441x}{400}+\frac{441x}{8000}=\frac{9261x}{8000}$, amount at the end of the third year.

Therefore, $$\frac{9261x}{8000}-15.25=2300.$$

Clearing of fractions, $$9261x-122000=18400000.$$

Uniting terms and dividing, $$x=2000.$$

Ex. (38.) Let x = the sum left by his father.

Then, $$x-\frac{x}{5}=\frac{4x}{5}, \text{ sum remaining.}$$

$$\frac{4x}{5}+124= \text{ his new estate.}$$

Then, $\frac{2}{3}\left(\frac{4x}{5}+124\right)=\frac{8x}{15}+\frac{248}{3}=$ the sum lost.

$\left(\frac{4x}{5}+124\right)-\left(\frac{8x}{15}+\frac{248}{3}\right)=\frac{4x}{15}+\frac{124}{3}=$ estate remaining.

$$\frac{4x}{15}+\frac{124}{3}+274+86=\frac{x}{2}.$$

Collecting terms, &c., $12040=7x.$

Changing terms, &c., $x=1720.$

Ex. (39.) Let $x=$ the money A had.

Then, $\frac{5x-60}{5}=\frac{x}{2}.$

Clearing of fractions, $10x-120=5x.$

Transposing, &c., $5x=120.$

Dividing, $x=24.$

Ex. (40.) Let $x=$ the sum he gave the youngest son.

Then, $x+133=$ the sum he gave the second son.

And $2x+133=$ the sum he gave the oldest.

Hence, $x+x+133+2x+133=1862.$

Collecting, &c., $4x=1862-266.$

Reducing, $4x=1596.$

Dividing, $x=399$, youngest son

$399+133=532$, second son.

$399+532=931$, oldest son.

Hence he gave the youngest son \$399, the second \$532, and the oldest \$931.

Ex. (41.) Let $x=$ the dollars the purse contained.

Then, $\frac{x}{2}-15=$ A's share.

And $\frac{x}{4}+13=$ B's share.

$27=$ C's share.

Therefore, $\frac{x}{2}-15+\frac{x}{4}+13+27=x.$

Clearing of fractions, $2x-60+x+52+108=4x.$

Collecting, &c., $x=100.$

Ex. (42.) Let x = the sum lent.

Then, $x+\frac{3x}{7}=500.$

Clearing of fractions, $7x+3x=3500.$

Collecting terms, $10x=3500.$

Dividing, $x=350.$

Ex. (43.) Let x = the value of the estate.

Then, $x+\frac{x}{4}-760+600=2000.$

Clearing of fractions, $4x+x-3040+2400=8000.$

Collecting terms, &c., $5x=8640.$

Dividing, $x=1728.$

Ex. (44.) Let x = John's shillings.

And $x+40$ = James' shillings.

Then, $\frac{x}{4}=\frac{x+40}{9}.$

Clearing of fractions, $9x=4x+160.$

Transposing, &c., $5x=160.$

Dividing, $x=32$, John's.

$32+40=72$, James'.

Ex. (45.) Let x = the barrels bought.

Then, $\frac{x}{2}+4$ = barrels sold to A.

And $x-\left(\frac{x}{2}+4\right)=\frac{x}{2}-4$ = barrels remaining.

And $\left(\frac{x}{2}-4\right)-\left[\frac{3}{4}\left(\frac{x}{2}-4\right)-4\right]=20.$

Subtracting, $\frac{x}{8}-1+4=20.$

Clearing of fractions, $x-8+32=160.$

Uniting terms, &c., $x=136$ barrels.

Ex. (46.) Let x = the number.

Then, $\frac{x-7}{6}=5.$

Clearing of fractions, $x-7=30.$

$x=37.$

Ex. (47.) Let x = the greater.
And $44-x$ = the less.

Then, $\frac{3x}{4}-6=\frac{3}{5}(44-x)$.

Clearing of fractions, $15x-120=528-12x$.
Uniting terms, $27x=648$.
Dividing, $x=24$, the greater.
$44-24=20$, the less.

Ex. (48.) Let x = the greater number.
Then, $43-x$ = the less number.

Therefore, $17-(43-x)=\frac{x-20}{3}$.

Subtracting, $-26+x=\frac{x-20}{3}$.

Clearing of fractions, $-78+3x=x-20$.
Uniting terms, &c., $2x=58$.
Dividing, $x=29$, the greater.
$43-29=14$, the less.

Ex. (49.) Let x = the time the son would reap the field; then, as Jones could reap the field in 10 days, it is evident he could reap $\frac{1}{10}$ of it in one day. And, as he with his son could reap the field in 8 days, they would both together reap $\frac{1}{8}$ of it in one day.

Therefore, $\frac{1}{8}-\frac{1}{10}=\frac{1}{40}$ field : 1 field : : 1 day : x days.

Multiplying extremes, &c., $\frac{x}{40}=1$.

Clearing of fractions, $x=40$ days.

Ex. (50.) Let x = the number of shillings A must pay B. Then, as A could reap $\frac{1}{9}$ of the field in one day, and A and B $\frac{1}{4}$ of it in a day, it is evident B could reap $\frac{1}{4}-\frac{1}{9}=\frac{5}{36}$ of it in one day, and four times $\frac{5}{36}$ of it in 4 days, $=\frac{5}{9}$ of the field.

Therefore, $1 : \frac{5}{9} : : 90 : x$
$x=50$ shillings.

Ex. (51.) Let $x=$ the value of the first horse.

Then, $\frac{5}{3}(x+30)=\frac{5x+150}{3}=$ second horse.

Therefore, $\frac{5x+150}{3}+30=3x$.

Clearing of fractions, $5x+150+90=9x$.

Uniting terms, &c., $-4x=-240$.

Dividing by -4, $x=60$, first horse.

$\frac{5}{3}(60+30)=150$, second horse.

Hence the first horse is worth \$60, and the second \$150.

Ex. (52.) Let $x=$ the whole sum lent.

Then, $\frac{3x}{8}\times.05+\frac{5x}{8}\times.06=180$.

Clearing of fractions, &c., $.15x+.3x=1440$.

Uniting terms, $.45x=1440$.

Dividing by .45, $x=3200$.

$\frac{3}{8}(3200)=1200$, at 5 per cent.

$3200-1200=2000$, at 6 per cent.

Ex. (53.) Let $x=$ the time A and B could do the work; then, as A could do it in 12 days, and B in 10 days, A would do $\frac{1}{12}$ and B $\frac{1}{10}$ of it in one day.

Therefore, $\frac{1}{12}+\frac{1}{10}=\frac{11}{60}:1::1$ day : x days.

Multiplying extremes, &c., $\frac{11x}{60}=1$.

Clearing of fractions, $11x=60$.

Dividing, $x=5\frac{5}{11}$ days, A and B.

Again, as A would do $\frac{1}{12}$ of it in one day, and C $\frac{1}{8}$ of it in a day;

Therefore, $\frac{1}{12}+\frac{1}{8}=\frac{5}{24}$ work : 1 work :: 1 day : x days.

Multiplying extremes, &c., $\frac{5x}{24}=1$.

Clearing of fractions, $5x=24$.

Dividing, $x=4\frac{4}{5}$, A and C.

Again, let x = the time B and C would do the work. As B would do $\frac{1}{10}$ of it in one day, and C $\frac{1}{8}$ of it;

Therefore, $\frac{1}{10}+\frac{1}{8}=\frac{9}{40}$ work : 1 work : : 1 day : x days.

Multiplying extremes, &c., $\frac{9x}{40}=1.$

Clearing of fractions, $9x=40.$

Dividing, $x=4\frac{4}{9}$, B and C.

Again, let x = the time A, B and C would do the work. As A would do $\frac{1}{12}$ of the labor in one day, B $\frac{1}{10}$, and C $\frac{1}{8}$ of it;

Therefore, $\frac{1}{12}+\frac{1}{10}+\frac{1}{8}=\frac{37}{120}$ work : 1 work : : 1 day : x days.

Multiplying extremes, &c., $\frac{37x}{120}=1.$

Clearing of fractions, $37x=120.$

Dividing, $x=3\frac{9}{37}.$

Ex. (54.) Let x = the principal.

Then, $x+x\times.10\times4=780+780\times.06\times5.$

Collecting, &c., $1.4x=1014.$

Dividing, $x=724.28\frac{4}{7}.$

Ex. (55.) Let x = the time.

Then, $500\times8\times x=270\times4\times6.$

Multiplying, $4000x=6480.$

Dividing, $x=1\frac{31}{50}$ years.

Ex. (56.) Let $6x$ = the number of leaps the greyhound must take.

Therefore, $9x$ = the number the fox takes at the same time.

And $9x+60$ = the whole number the fox takes.

Therefore, $3 : 7 :: 6x : 9x+60.$

Multiplying extremes, &c., $27x+180=42x.$

Changing terms, &c., $15x=180.$

Dividing, $x=12.$

Multiplying, $6x=72$, greyhound.

" $9x=108$, the fox.

That is, the greyhound makes 72 leaps and the fox 108.

Ex. (57.) Let $5x$ = the number of dollars the men received.
Then, $46-5x$ = the number the women received.
And x = the sum one man received.
And $8-x$ = the sum one woman received.

Then, $56-7x=46-5x$.
By transposition, $2x=10$.
Dividing, $x=5$.
Multiplying, $5x=25$, men received.
And $46-25=21$, women received.

Ex. (58.) Let x = the value of the first farm.

Then, $\frac{7}{9}(183+x)=\frac{1281+7x}{9}$ = value of the second farm.

And $\frac{1281+7x}{9}+183=\frac{13x}{8}$.

Clearing of fractions, $10248+56x+13176=117x$.
Transposing and uniting, $61x=23424$.
Dividing, $x=384$, 1st farm.
$\frac{7}{9}(183+384)=441$, 2nd farm.

Ex. (59.) 1st. To find the position of the second-hand.

Let x = the time and place of the minute-hand.

Then, as the second-hand moves sixty times as fast as the minute-hand, it is evident it must have made one revolution of sixty seconds, and also sixty times x, $= 60x$, in order to be between the hour-hand and minute-hand, and be equal distances between them; and, as the seconds in any given distance are sixty more in number than the minutes, it is evident that

$60x-60=$ the time and place of the second-hand.

And, as the hour-hand moves only $\frac{1}{12}$ as fast as the minute-hand,

Then, $\frac{x}{12}$ = the time and place of the hour-hand.

Now, by the question, the difference in time between the minute-hand and second-hand is equal to the time between the second-hand and hour-hand

Therefore, $x-\overline{60x-60}=\overline{60x-60}-\frac{x}{12}$.

Clearing of fractions, $1427x=1440$ m.$=86400$ seconds.

Dividing, $x=60\frac{780}{1427}$ seconds.

2nd. To find the position of the minute-hand.

Again, let $x =$ the time and place of the minute-hand. Then it is evident the second-hand will have made one revolution of 60 seconds, and will also have passed over sixty times as many seconds as the minute-hand has minutes.

Therefore, $60x-60=$ the time and place of the second-hand. And, as the hour-hand, as we have before stated, moves $\frac{1}{12}$ as fast as the minute-hand,

Then, $\frac{x}{12}=$ the time and place of the hour-hand.

As the space or time between the minute-hand and second-hand, by the question, is equal to the time between the minute-hand and hour-hand,

Therefore, $\overline{60x-60}-x=x-\frac{x}{12}$.

Clearing of fractions, $720x-720-12x=12x-x$.

Uniting terms, $697x=720$ m.$=43200$ seconds.

Dividing, $x=61\frac{683}{697}$ seconds.

3d. To find the position of the hour-hand.

Let $x =$ the time and place of the minute-hand.

$\frac{x}{12}=$ the time and place of the hour-hand.

And $60-60x=$ the time and place of the second-hand.

It is evident, in this last case, that the second-hand has not performed one revolution.

Therefore, $x-\frac{x}{12}=\frac{x}{12}+\overline{60-60x}$.

Clearing of fractions, $12x-x=x+720-720x$.

Uniting terms, $730x=720$ m.$=43200$ seconds

Dividing, $x=59\frac{13}{73}$ seconds.

Ex. (60.) Let $x=$ the number.
Then, $3x+12-54=144-3x.$
Collecting terms, $6x=144-12+54=186.$
$x=31.$

SIMPLE EQUATIONS OF THE FIRST DEGREE, CONTAINING TWO UNKNOWN TERMS.

ART. 152. (p. 93.)

Ex. (6.) 1. Conditions, $3x+7y=33.$
2. Conditions, $2x+4y=20.$
3. Multiplying (1) by 2, $6x+14y=66.$
4. Multiplying (2) by 3, $6x+12y=60.$
5. Subtracting, $2y=6.$
6. Dividing, $y=3.$
7. Multiplying (1) by 4, $12x+28y=132.$
8. Multiplying (2) by 7, $14x+28y=140.$
9. Subtracting, $2x=8.$
10. Dividing, $x=4.$

Hence $x=4$, and $y=3$.

Ex. (7.) 1. Conditions, $7x+2y=39.$
2. Conditions, $3x-4y=7.$
3. Multiplying (1) by 3, $21x+6y=117.$
4. Multiplying (2) by 7, $21x-28y=49.$
5. Subtracting, $34y=68.$
6. Dividing, $y=2.$
7. Multiplying (1) by 2, $14x+4y=78.$
8. Adding (2) to (7), $17x=85.$
9. Dividing, $x=5.$

Hence $x=5$, and $y=2$.

Ex. (8.) 1. Conditions, $6x-3y=27.$
2. Conditions, $4x-6y=-2.$
3. Multiplying (1) by 2, $12x-6y=54.$
4. Multiplying (2) by 3, $12x-18y=-6.$
5. Subtracting, $12y=60.$
6. Dividing, $y=5.$

7. Multiplying (1) by 2, $12x-6y=54.$
8. The (2), $4x-6y=-2.$
9. Subtracting (8) from (7), $8x=56.$
10. Dividing, $x=7.$

Hence $x=7$, and $y=5$.

Ex. (9.) 1. Conditions, $7x+3y=62.$
2. Conditions, $5x-2y=36.$
3. Multiplying (1) by 2, $14x+6y=124.$
4. Multiplying (2) by 3, $15x-6y=108.$
5. Adding (3) to (4), $29x=232.$
6. Dividing, $x=8.$
7. Multiplying (1) by 5, $35x+15y=310.$
8. Multiplying (2) by 7, $35x-14y=252.$
9. Subtracting (8) from (7), $29y=58.$
10. Dividing, $y=2.$

Hence $x=8$, and $y=2$.

Ex. (10.) 1. Conditions, $12x+8y=116.$
2. Conditions, $2x-y=3.$
3. Multiplying (2) by 6, $12x-6y=18.$
4. Subtracting (3) from (1), $14y=98.$
5. Dividing, $y=7.$
6. Multiplying (2) by 8, $16x-8y=24.$
7. Adding (1) to (6), $28x=140.$
8. Dividing, $x=5.$

Hence $x=5$, and $y=7$.

Ex. (11.) 1. Conditions, $11x+3y=124.$
2. Conditions, $2x-6y=-56.$
3. Multiplying (1) by 2, $22x+6y=248.$
4. Adding (2) to (3), $24x=192.$
5. Dividing, $x=8.$
6. Multiplying (2) by 11, $22x-66y=-616.$
7. Subtracting (6) from (3), $72y=864.$
8. Dividing, $y=12.$

Hence $x=8$, and $y=12$.

Ex. (12.) 1. Conditions, $9x+4y=58.$
2. Conditions, $3x+2y=26.$
3. Multiplying (2) by 3, $9x+6y=78.$
4. Subtracting (1) from (3), $2y=20.$
5. Dividing, $y=10.$
6. Multiplying (2) by 2, $6x+4y=52.$
7. Subtracting (6) from (1), $3x=6.$
8. Dividing, $x=2.$

Hence $x=2$, and $y=10$.

Ex. (13.) 1. Conditions, $6x+5y=112.$
2. Conditions, $8x-2y=80.$
3. Multiplying (1) by 2, $12x+10y=224.$
4. Multiplying (2) by 5, $40x-10y=400.$
5. Adding (3) to (4), $52x=624.$
6. Dividing, $x=12.$
7. Multiplying (1) by 4, $24x+20y=448.$
8. Multiplying (2) by 3, $24x-6y=240.$
9. Subtracting (8) from (7), $26y=208.$
10. Dividing, $y=8.$

Hence $x=12$, and $y=8$.

Ex. (14.) 1. Conditions, $7x-2y=-6.$
2. Conditions, $2x+2y=24.$
3. Adding (1) to (2), $9x=18.$
4. Dividing, $x=2.$
5. Multiplying (4) by 7, $7x=14.$
6. Subtracting (1) from (5), $2y=20.$
7. Dividing, $y=10.$

Hence $x=2$, and $y=10$.

Ex. (15.) 1. Conditions, $6x+11y=115.$
2. Conditions, $8x-22y=-30.$
3. Multiplying (1) by 2, $12x+22y=230.$
4. Adding (2) to (3), $20x=200.$
5. Dividing, $x=10.$

6. Multiplying (1) by 2, $12x+22y=230$.
7. Multiplying (5) by 12, $12x=120$.
8. Subtracting (7) from (6), $22y=110$.
9. Dividing, $y=5$.

Hence $x=10$, and $y=5$.

Ex. (16.) 1. Conditions, $2x+3y=47$.
2. Conditions, $10x-12y=-62$.
3. Multiplying (1) by 4, $8x+12y=188$.
4. Adding (2) to (3), $18x=126$.
5. Dividing, $x=7$.
6. Multiplying (1) by 5, $10x+15y=235$.
7. Subtracting (2) from (6), $27y=297$.
8. Dividing, $y=11$.

Hence $x=7$, and $y=11$.

Ex. (17.) 1. Conditions, $\frac{x}{2}-\frac{y}{3}=0$.
2. Conditions, $\frac{x}{4}+\frac{y}{6}=6$.
3. Clearing of fractions, $3x-2y=0$.
4. " " " $3x+2y=72$.
5. Adding (3) to (4), $6x=72$.
6. Dividing, $x=12$.
7. Subtracting (3) from (4), $4y=72$.
8. Dividing, $y=18$.

Hence the value of $x=12$, and $y=18$.

Ex. (18.) 1. Conditions, $\frac{3x}{5}-y-11$.
2. Conditions, $x+\frac{y}{5}=37$.
3. Clearing of fractions, $3x-5y=55$.
4. " " and multiplying by 5, $25x+5y=925$.
5. Adding (3) to (4), $28x=980$.
6. Dividing, $x=35$.

7. Multiplying (3) by 5, $15x-25y=275.$
8. Multiplying (2) by 15, $15x+3y=555.$
9. Subtracting (7) from (8), $28y=280$
10. Dividing, $y=10.$

Hence the values of $x=35$, and $y=10$.

Ex. (19.) 1. Conditions, $\frac{4x}{7}-\frac{2y}{3}=2.$
2. Conditions, $x+7y=175.$
3. Multiplying (1) by 21, $12x-14y=42.$
4. Multiplying (2) by 2, $2x+14y=350.$
5. Adding (3) to (4), $14x=392.$
6. Dividing, $x=28.$
7. Multiplying (2) by 12, $12x+84y=2100.$
8. Subtracting (3) from (7), $98y=2058.$
9. Dividing, $y=21$

Hence the values of $x=28$, and $y=21$.

Ex. (20.) Given $\left\{\begin{array}{l}\frac{7x}{8}-\frac{y}{9}=19\\ 3x+3y=126\end{array}\right\}$ to find the value of x and y.

1. By the first condition, $\frac{7x}{8}-\frac{y}{9}=19.$
2. By the second condition, $3x+3y=126.$
3. Multiplying the 1st by 72, $63x-8y=1368.$
4. Multiplying the 2d by 21, $63x+63y=2646.$
5. Subtracting the 3d from the 4th, $71y=1278.$
6. Dividing, $y=18.$
7. Substituting 18 for y in the 3d, $63x-144=1368.$
8. Transposing, &c., $63x=1512.$
9. Dividing, $x=24.$

Ex. (21.) Given $\left\{\begin{array}{l}14x+\frac{5y}{6}=38\\ x+12y=146\end{array}\right\}$ to find the value of x and y.

1. By the first condition, $14x+\frac{5y}{6}=38.$
2. By the second condition, $x+12y=146$
3. By transposition, $x=146-12y.$
4. By substituting the value of x in the 3d,

$$14(146-12y)+\frac{5y}{6}=38.$$

5. Reducing, &c., $2044-168y+\frac{5y}{6}=38.$
6. Clearing of fractions, $12264-1008y+5y=228.$
7. Transposing, &c., $1003y=12036.$
8. Dividing, $y=12.$
9. By substitution, $x=146-144=2.$

Ex. (22.) Given $\left\{\begin{matrix}\frac{x}{7}-\frac{7y}{10}=-20\\ \frac{x}{4}+3y=134\end{matrix}\right\}$ to find the value of x and y.

1. First condition, $\frac{x}{7}-\frac{7y}{10}=-20.$
2. Second condition, $\frac{x}{4}+3y=134.$
3. Multiplying (1) by 70, $10x-49y=-1400.$
4. Multiplying (2) by 40, $10x+120y=5360.$
5. Subtracting (3) from (4), $169y=6760.$
6. Dividing, $y=40.$
7. By substituting the value of y in (4), $10x+4800=5360.$
8. Transposing, &c., $x=56.$

Ex. (23). Given $\left\{\begin{matrix}ax+by=c\\ mx+ny=d\end{matrix}\right\}$ to find the value of x and y.

1. First condition, $ax+by=c.$
2. Second condition, $mx+ny=d.$
3. Multiplying (2) by a, $max+any=ad.$
4. Multiplying (1) by m, $max+mby=mc.$
5. Subtracting (4) from (3), $any-mby=ad-mc.$
6. Dividing, $y=\frac{ad-mc}{an-mb}.$

7. Multiplying (2) by b, $bmx+bny=bd.$
8. Multiplying (1) by n, $anx+bny=cn.$
9. Subtracting (8) from (7), $bmx-anx=bd-cn.$
10. Dividing, $x=\frac{bd-cn}{bm-an}.$

Ex. (24.) Given $\left\{\begin{array}{l}\frac{x}{a}-\frac{y}{b}=m\\ \frac{x}{c}+\frac{y}{d}=n\end{array}\right\}$ to find the value of x and y.

1. First condition, $\frac{x}{a}-\frac{y}{b}=m.$
2. Second condition, $\frac{x}{c}+\frac{y}{d}=n.$
3. Clearing (1) of fractions, $bx-ay=abm.$
4. Clearing (2) of fractions, $dx+cy=cdn.$
5. Multiplying (3) by d, $bdx-ady=abdm.$
6. Multiplying (4) by b, $bdx+bcy=bcdn.$
7. Subtracting (5) from (6), $ady+bcy=bcdn-abdm.$
8. Dividing, $y=\frac{bcdn-abdn}{ad+bc}.$
9. Multiplying (3) by c, $bcx-acy=abcm.$
10. Multiplying (4) by a, $adx+acy=acdn.$
11. Adding (9) and (10), $adx+bcx=abcm+acdn.$
12. Dividing, $x=\frac{abcm+acdn}{ad+bc}.$

Ex. (25.) Given $\left\{\begin{array}{l}\frac{x}{2}-12=\frac{y}{4}+8\\ \frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27\end{array}\right\}$ to find the value of x and y.

1. First condition, $\frac{x}{2}-12=\frac{y}{4}+8.$
2. Second condition, $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27.$
3. Clearing (1) of fractions, &c., $2x-y=\ 80.$

4. Clearing (2) of fractions, &c., $47x-18y=2100$.
5. Multiplying (3) by 18, $36x-18y=1440$.
6. Subtracting (5) from (4), $11x=660$.
7. Dividing, $x=60$.
8. Substituting the value of x in (3), $120-y=80$.
9. Transposing, &c., $y=40$.

SIMPLE EQUATIONS, CONTAINING TWO OR MORE UNKNOWN TERMS.

ART. 152. (p. 97.)

Ex. (3.) Given $\begin{array}{r}1\\2\\3\end{array}\left\{\begin{array}{r}3x-y-2z=0\\6x+2y+3z=45\\4x+3y-z=31\end{array}\right\}$ to find the value of x, y and z.

Subtracting twice the first from the second, and we have

4. $4y+7z=45$.

Subtracting twice the second from three times the third,

5. $5y-9z=3$.

Subtracting four times the fifth from five times the fourth,

6. $71z=213$.
7. $z=3$.

Substituting for z its value in the fourth,

8. $4y+21=45$.
9. $y=6$.

Substituting for y and z their values in the third,

10. $4x+18-3=31$.
11. $x=4$.

Ex. (4.) Given $\begin{array}{r}1\\2\\3\end{array}\left\{\begin{array}{r}8x-9y-7z=-36\\12x-y-3z=36\\6x-2y-z=10\end{array}\right\}$ to find the values of x, y and z.

Subtracting three times (1) from twice (2),

4. $25y+15z=180$.
5. $5y+3z=36$.

Subtracting four times (3) from twice (2),

6. $6y-2z=32$.

Subtracting five times (6) from six times (5),

7. $28z=56.$

8. $z=2.$

Substituting for z its value in the (6),

9. $6y-4=32.$

10. $6y=36.$

11. $y=6.$

Substituting for y and z their values in the (3),

12. $6x-12-2=10.$

13. $6x=24.$

14. $x=4.$

Ex. (5.) Given $\begin{matrix}1\\2\\3\end{matrix}\left\{\begin{aligned}7x+4y-z&=78\\4x-5y-3z&=-21\\x-3y-4z&=-37\end{aligned}\right\}$ to find the values of x, y and z.

Subtracting seven times (2) from four times (1),

4. $51y+17z=459.$

Subtracting four times (3) from (2),

5. $7y+13z=127.$

Subtracting seven times (4) from fifty-one times (5),

6. $544z=3264.$

7. $z=6.$

Substituting for z its value in the (5),

8. $7y+78=127.$

9. $7y=49.$

10. $y=7.$

Substituting for y and z their values in (1),

11. $7x+28-6=78.$

12. $7x=56.$

13. $x=8.$

Ex. (6.) Given $\begin{matrix}1\\2\\3\end{matrix}\left\{\begin{aligned}x+y&=30\\x+z&=25\\y+z&=15\end{aligned}\right\}$ to find the values of x, y and z.

Subtracting (2) from (1),

4. $y-z=5.$

Subtracting (4) from (3),

5. $2z=10.$
6. $z=5.$

Substituting for z its value in (3),

7. $y+5=15.$
8. $y=10.$

Substituting for y its value in the (1),

9. $x+10=30.$
10. $x=20.$

Ex. (7.) Given $\begin{matrix}1\\2\\3\end{matrix}\left\{\begin{array}{l}8x-4y=24-z\\6x+y=z+84\\x+80=3y+4z\end{array}\right\}$ to find the values of x, y and z.

Subtracting three times (1) from four times (2),

4. $16y-7z=264.$

Subtracting (2) from six times (3),

5. $19y+23z=564.$

Subtracting nineteen times (4) from sixteen times (5),

6. $501z=4008.$
7. $z=8.$

Substituting for z its value in (4),

8. $16y=264+56=320.$
9. $y=20.$

Substituting for y and z their values in (1),

10. $8x-80=24-8.$
11. $8x=96.$
12. $x=12.$

Ex. (8.) Given $\begin{matrix}1\\2\\3\end{matrix}\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}-\frac{z}{4}=23\\\frac{x}{3}-\frac{y}{4}+\frac{z}{2}=12\\\frac{x}{4}+\frac{y}{2}-\frac{z}{3}=17\end{array}\right\}$ to find the values of x, y and z.

4. Clearing of fractions, $6x+4y-3z=276.$
5. Clearing of fractions, $4x-3y+6z=144.$

4. Clearing of fractions, $3x+6y-4z=204.$

Subtracting three times (4) from twice (5),

7. $17y-24z=120.$

Subtracting three times (5) from four times (6),

8. $33y-34z=384.$

Subtracting thirty-three times (7) from seventeen times (8),

9. $214z=2568.$

10. $z=12.$

Substituting for z its value in (7),

11. $17y-288=120.$

12. $y=24.$

Substituting for y and z their values in (1),

13. $6x+96-36=276.$

14. Uniting terms, $6x=216.$

$x=36.$

Ex. (9.) Given
$$\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \end{array}\left\{\begin{array}{l} 3u+x+2y-z=22 \\ \quad 4x-y+3z=35 \\ 4u+3x-2y=19 \\ 2u+4y+2z=46 \end{array}\right\}$$
to find the values of u, x, y, z.

Subtract three times (3) from four times (1),

5. $-5x+14y-4z=31.$

Subtract the (3) from twice (4),

6. $-3x+10y+4z=73.$

Add three times (2) to four times (6),

7. $37y+25z=397.$

Add five times (2) to four times (5),

8. $51y-z=299.$

Subtract thirty-seven times (8) from fifty-one times (7),

9. $1312z=9184.$

10. $z=7.$

Substituting for z its value in the (8),

11. $51y-7=299.$

12. Uniting terms, $51y=306.$

13. $y=6.$

Substituting for y and z their values in (2),
14. $4x-6+21=35.$
15. Uniting terms, $4x=20.$
16. $x=5.$

Substituting for y and z their values in (3),
17. $4u+15-12=19.$
18. Uniting terms, $4u=16.$
19. $u=4.$

EQUATIONS OF THE FIRST DEGREE, CONTAINING SEVERAL UNKNOWN QUANTITIES.

PROBLEMS. (p. 98.)

Ex. (1.) 1. Conditions, $A+\frac{B+C}{2}=55.$

2. Conditions, $B+\frac{A+C}{3}=50.$

3. " $C+\frac{A+B}{5}=50.$

Clearing of fractions,
4. $2A+B+C=110.$
5. $A+3B+C=150.$
6. $A+B+5C=250.$
7. Subtracting (4) from twice (5), $5B+C=190.$
8. Subtracting (5) from (6), $-2B+4C=100.$
9. Adding twice (7) to five times (8), $22C=880.$
10. Dividing, $C=40.$
11. Substituting for C its value in (7), $5B+40=190.$
12. Reducing, $5B=150.$
13. Dividing, $B=30$
14. Substituting for B and C their values in (5),
$A+90+40=150.$
15. Collecting terms, $A=20$

Hence A had \$20, B \$30, and C \$40.

Ex. (2.) Let the prices of the sugar $= x, y$ and z.

1. Then, $3x+4y+2z= 60.$
2. " $4x+y+5z= 59.$
3. And $x+10y+3z= 90.$
4. Subtracting three times (2) from four times (1), $13y-7z= 63.$
5. Subtracting (2) from four times (3), $39y+7z=301.$
6. Adding (4) and (5), $52y=364.$
7. Dividing, $y= 7.$
8. Substituting for y its value in (4), $91-7z= 63.$
9. Transposing, $7z= 28.$
10. Dividing, $z= 4.$
11. Substituting for y and z their values in (3), $x+70+12= 90.$
12. Uniting terms, $x= 8.$

Hence the price of the first quality is 8 cents per lb.; the second, 7 cents; the third, 4 cents.

Ex. (3). Let $x =$ best horse, $y =$ the worst horse, and z $=$ the harness.

1. Then, $x+y+z=120.$
2. " $y+z= 2x.$
3. And $x+z= 3y.$
4. Subtracting (2) from (1), $x=120-2x.$
5. Transposing, &c., $x= 40.$
6. Subtracting (3) from (1), $y=120-3y.$
7. Transposing, &c., $y= 30.$
8. Substituting the values of x and y in (1), $40+30+z=120.$
9. Transposing, &c., $z= 50.$

Hence the value of worst horse is \$30, the best horse \$40, and the harness \$50.

Ex. (4.) Let x, y and $z =$ the three numbers.

1. Then, $x+\frac{y+z}{2}=34.$

2. Then, $y+\frac{x+z}{3}=34.$
3. And $z+\frac{x+y}{4}=34.$

Clearing of fractions, we have

4. $2x+y+z=68.$
5. $x+3y+z=102.$
6. $x+y+4z=136.$
7. Subtracting (4) from twice (5), $5y+z=136.$
8. Subtracting (5) from (6), $-2y+3z=34.$
9. Adding twice (7) to five times (8), $17z=442.$
10. Dividing, $z=26.$
11. Substituting for z its value in (7), $5y+26=136.$
12. Transposing, $5y=110.$
13. Dividing, $y=22.$
14. Substituting for y and z their value in (6),
$x+22+104=136.$
15. Transposing, &c., $x=10.$

Hence the numbers are 10, 22 and 26.

Ex. (5.) Let the three digits be represented by $(x-y)$, x, and $(x+y)$; then the number will be $100(x-y)+10x+x+y=111x-99y$.

1. Therefore, $\frac{111x-99y}{\frac{3x}{2}}=41.$
2. Clearing of fractions, $74x-66y=41x.$
3. Transposing, $33x=66y.$
4. Dividing, $x=2y.$
5. Again, $111x-99y+396=100(x+y)+10x+x-y.$
6. $111x-99y+396=100x+100y+10x+x-y.$
7. Reducing, $198y=396.$
8. Dividing, $y=2.$
9. Substituting for y its value in (4),
$x=4.$

$4-2=2$, first digit; 4, second digit; and $4+2=6$, the third digit; and the number is 246.

Ex. (6.) Let x = bushels of wheat, and y = bushels of rye.

1. Then, $7x=4y+3$.
2. And $x : y :: 3 : 5$.
3. Multiplying extremes, &c., $5x=3y$.
4. Multiplying (1) by 5, $35x=20y+15$.
5. Multiplying (3) by 7, $35x=21y$.
6. Subtracting (5) from (4), &c., $y=15$, rye.
7. Substituting for y its value in (3), $5x=45$.
8. Dividing, $x=9$, wheat.

Hence there are 9 bushels of wheat, and 15 of rye.

Ex. (7.) Let x = the property of A, and y = the property of B.

1. Then, $7x+\frac{y}{7}=990$.
2. And $7y+\frac{x}{7}=510$.

Clearing the terms of fractions, we have

3. $49x+y=6930$.
4. And $x+49y=3570$.
5. Subtracting (3) from forty-nine times (4),
$2400y=168000$.
6. Dividing, $y=70$.
7. Substituting for y its value in (3), $49x+70=6930$.
8. Transposing and dividing, $x=140$.

Hence A's property is \$140, and B's \$70.

Ex. (8.) Let x = A's age, and y = B's age.

1. Then, $y-\frac{x}{7}+5=6$.
2. And $\frac{y}{5}+4=\frac{x}{14}$.
3. Clearing of fractions, $7y-x+35=42$.
4. " " " $14y+280=5x$.
5. Subtracting twice (3) from (4), $3x=294$.
6. Dividing. $x=98$.

7. Substituting for x its value in (3), $7y-98+35=42.$
8. Reducing terms, $7y=105.$
9. Dividing, $y=15$

Hence A's age $=$ 98 years, and B's age $=$ 15 years.

Ex. (9.) Let $\frac{x}{y}=$ the fraction.

1. Then, $\frac{x+1}{y}=\frac{1}{3}.$
2. And $\frac{x}{y+1}=\frac{1}{4}.$
3. Clearing of fractions, $3x+3=y.$
4. " " " $4x=y+1.$
5. Subtracting (3) from (4), $x-3=1.$
6. Uniting terms, &c., $x=4.$
7. Substituting for x its value in (3), $12+3=y.$
8. Transposing, &c., $y=15.$

Hence the fraction is $\frac{4}{15}$.

Ex. (10.) Let $x=$ A's age, and $y=$ B's age.

1. Then, $x-\frac{x-y}{2}=25.$
2. And $y-\frac{x+y}{5}=\frac{x}{3}.$
3. Clearing of fractions, $2x-x+y=50.$
4. " " " $15y-3x-3y=5x.$
5. Adding one times (4) to eight times (3), $20y=400.$
6. Dividing, $y=20.$
7. Substituting for y its value in (3), $x+20=50.$
8. Transposing, &c., $x=30.$

Hence A's age $=$ 30 years, and B's $=$ 20 years.

Ex. (11.) Let $x=$ the larger number, and $y=$ the less.

1. Conditions, $4(x+y)-\frac{x-y}{4}=62.$
2. " $(x+y)-(x-y)=\frac{2x}{3}.$

3. Clearing of fractions, $16x+16y-x+y=248.$
4. Uniting terms, $15x+17y=248.$
5. Uniting terms of (2), $2y=\frac{2x}{3}.$
6. Clearing of fractions, $6y=2x.$
7. Adding fifteen times (6) to twice (4), $124y=496.$
8. Dividing, $y=4.$
9. Substituting for y its value in (6), $2x=24.$
10. Dividing, $x=12.$

Therefore, 12 = the larger number, and 4 = the less.

Ex. (12.) Let x = A's money, y = B's money, and z = C's money.

1. Then, by conditions, $x+100=y+z.$
2. " " " $y+100=2(x+z).$
3. " " " $z+100=3(x+y).$
4. Adding twice (1) to (2), $300=y+4z.$
5. Adding three times (1) to (3), $400=6y+2z.$
6. Subtracting half (5) from thrice (4), $700=11z.$
7. Transposing and dividing, $z=63\frac{7}{11}.$
8. Substituting for z its value in (5), $400=6y+127\frac{3}{11}.$
9. Transposing, &c., $6y=272\frac{8}{11}.$
10. Dividing, $y=45\frac{5}{11}.$
11. Substituting for y and z their value in (1),
$$x+100=45\tfrac{5}{11}+63\tfrac{7}{11}.$$
12. Uniting terms, &c., $x=9\frac{1}{11}.$

Hence, first man's money, $\$9\frac{1}{11}$; the second, $\$45\frac{5}{11}$; the third, $\$63\frac{7}{11}$.

Ex. (13.) Let x, y and z = their respective ages.

1. Then, by conditions, $x+y+z=90.$
2. " " " $(x+z)-y=30.$
3. " " " $(x+y)-z=\frac{z}{4}.$
4. Subtracting (2) from (1), $2y=60.$
5. Dividing, $y=30.$

6. Subtracting (3) from (2), $2z-2y=30-\frac{z}{4}$.
7. Multiplying (6) by 4, $8z-8y=120-z$.
8. Transposing, &c., (7), $9z-8y=120$.
9. Substituting for y its value in (8), $9z-240=120$.
10. Transposing, $9z=360$.
11. Dividing, $z=40$.
12. Substituting for y and z their values in (1),

$$x+30+40=90.$$

13. Collecting terms, &c., $x=20$.

Hence A's age 20, B's 30, and C's 40 years.

Ex. (14). Let w, x, y, z, represent their estates respectively.

1. Then, by conditions, $w+x+y+z=14000$.
2. “ “ “ $2w+3x+\frac{y}{2}+\frac{z}{5}=16000$.
3. “ “ “ $w+2x+2y+\frac{2z}{5}=18000$.
4. “ “ “ $\frac{w}{2}+\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=4000$.
5. Subtracting ten times (2) from twenty times (3),

$$10x+35y+6z=200000.$$

6. Subtracting ten times (2) from twenty times (1),

$$-10x+15y+18z=120000.$$

7. Subtracting sixty times (4) from thirty times (3),

$$40x+45y=300000.$$

8. Multiplying, (6) by 4, $-40x+60y+72z=480000$.
9. Adding (7) to (8), $105y+72z=780000$.
10. Adding (5) to (6), $50y+24z=320000$.
11. Subtracting (9) from three times (10), $45y=180000$.
12. Dividing, $y=4000$.
13. Substituting for y its value in (10),

$$200000+24z=320000.$$

14. Transposing, &c., $z=5000$.
15. Substituting for y its value in (7), $40x+180000=300000$.
16. Transposing, &c., $x=3000$.

17. Substituting for x, y, z, their values in (1), $w+3000+4000+5000=14000.$
18. Transposing, &c., $w=2000.$

Hence A's estate $2000, B's $3000, C's $4000, and D's $5000.

Ex. (15.) Let the four numbers be represented by w, x, y and z, respectively.

1. Then, by conditions, $w+\frac{x}{2}=357.$
2. " " " $x+\frac{y}{3}=476.$
3. " " " $y+\frac{z}{4}=595.$
4. " " " $\frac{w}{5}+z=714.$
5. Clearing of fractions, $2w+x=714.$
6. " " " $3x+y=1428.$
7. " " " $4y+z=2380.$
8. " " " $w+5z=3570.$
9. Subtracting (5) from twice (8), $-x+10z=6426.$
10. Adding (6) to three times (9), $y+30z=20706.$
11. Subtracting (7) from four times (10), $119z=80444.$
12. Dividing, $z=676.$
13. Substituting for z its value in (7), $4y+676=2380.$
14. Transposing, &c., $y=426.$
15. Substituting for y its value in (6), $3x+426=1428.$
16. Transposing, &c., $x=334.$
17. Substituting for x its value in (5), $2w+334=714.$
18. Transposing, &c., $w=190.$

Hence the four numbers are 190, 334, 426, and 676.

Ex. (16.) Let $x=$ the length of the field, and $y=$ its breadth.

1. Then, $x\times y=xy=$ the contents of it.
2. $\overline{x+5}\times\overline{y+4}=xy+5y+4x+20.$
3. By condition, $xy+240=xy+5y+4x+20.$
4. Transposing, &c., $5y+4x=220.$

5. Again, $\overline{x-4}\times\overline{y-5}=xy-4y-5x+20$.
6. Then, by conditions, $xy-210=xy-4y-5x+20$.
7. Transposing, &c., $4y+5x=230$.
8. The (4), $5y+4x=220$.
9. Multiplying (7) by 5, $20y+25x=1150$.
10. Multiplying (8) by 4, $20y+16x=880$.
11. Subtracting (10) from (9), $9x=270$.
12. Dividing, $x=30$.
13. Substituting x for its value in (8), $5y+120=220$.
14. Transposing, &c., $y=20$.

Therefore the length of the field is 30 rods, and its breadth 20 rods, and it contains 600 square rods.

Ex. (17.) Let $x =$ the price of wheat per bushel, in shillings, and $y =$ the price of the barley.

Then, $\frac{200}{x} =$ the number of bushels in the second offer.

1. Therefore, $12x=8y+56$.
2. And $\frac{200}{x}\times y=125$.
3. Therefore, $8y=5x$.
4. And $12x=5x+56$.
5. Transposing, $7x=56$.
6. And $x=8$.
7. Therefore, $y=5$.

The prices of wheat and barley per bushel were 8 and 5 shillings respectively.

Ex. (18.) Let $x =$ oxen, and $y =$ cows.

1. Conditions, $x+y=89$.
2. " $x-4-7=y-20$.
3. Uniting terms, $x-y=-9$.
4. Subtracting (3) from (1), $2y=98$.
5. Dividing. $y=49$.
6. Substituting for y its value in (1), $x+49=89$.
7. Transposing, &c., $x=40$.

There were 40 oxen and 49 cows.

Ex. (19.) Let x = A's turkeys, and y = B's turkeys.

1. Then, $x+5=y-5.$
2. And $x-15=\frac{3}{7}(y+15).$
3. Clearing of fractions, $7x-105=3y+45.$
4. Multiplying (1) by 7, &c., $7x+70=7y.$
5. Subtracting (4) from (3), &c., $4y=220.$
6. Dividing, $y=55.$
7. Substituting for y its value in (1), $x+5=55-5.$
8. Uniting terms, &c., $x=45.$

A had 45, and B 55 turkeys.

Ex. (20.) Let x = the larger, and y = the smaller number.

1. 1st condition, $\frac{x}{3}+\frac{y}{4}=25.$
2. 2d condition, $\frac{x}{4}-\frac{y}{6}=6.$
3. Clearing of fractions, and multiplying by 3, $12x+9y=900.$
4. " " " " 2, $12x-8y=288.$
5. Subtracting (4) from (3), $17y=612.$
6. Dividing, $y=36.$
7. Substituting for y its value in (3), $12x+324=900.$
8. Dividing, $x=48.$

Hence the numbers are 48 and 36.

Ex. (21.) Let x = the numerator, and y = the denominator.

1. Then, by conditions, $\frac{x+5}{y}=2.$
2. And " " $\frac{x}{y+2}=\frac{1}{2}.$
3. Clearing of fractions, $x+5=2y.$
4. " " " $2x=y+2.$
5. Subtracting (4) from twice (3), $10=3y-2.$
6. Connecting terms, $12=3y.$
7. Transposing, &c., $y=4.$
8. Substituting for y its value in (3), $x+5=8.$
9. Reducing, $x=3.$

Hence the fraction is $\frac{3}{4}$.

Ex. (22.) Let x = B's age, and y = C's age.

1. Then, by conditions, $\frac{x+3}{2}=y-3$.
2. And " " $x-3=y+3$.
3. Multiplying (1) by 2, $x+3=2y-6$.
4. Subtracting (2) from (3), $6=y-9$.
5. Transposing, &c., $y=15$.
6. Substituting for y its value in (2), $x-3=15+3$.
7. Transposing, &c., $x=21$.

Hence B's age 21, and C's age 15 years.

Ex. (23.) Let x = first number, and y = second number.

1. By first condition, $\frac{2x}{3}+\frac{3y}{4}=15\frac{2}{3}$.
2. By second condition, $\frac{3x}{4}-\frac{y}{7}=5\frac{11}{14}$.
3. Clearing of fractions, $8x+9y=188$.
4. " " " $21x-4y=162$.
5. Subtracting eight times (4) from twenty-one times (3), $221y=2652$.
6. Dividing, $y=12$.
7. Substituting for y its value in (3), $8x+108=188$.
8. Transposing, &c., $x=10$.

Hence the numbers are 10 and 12.

Ex. (24.) Let x = the larger part, and y = the smaller part.

1. Then, by conditions, $x+y=50$.
2. And " " $\frac{3x}{8}=\frac{2y}{3}$.
3. Clearing of fractions, &c., (2), $9x-16y=0$.
4. Nine times (1), $9x+9y=450$.
5. Subtracting (3) from (4,, $25y=450$.
6. Dividing, $y=18$.
7. Substituting for y its value in (1), $x+18=50$.
8. Transposing, &c., $x=32$.

Hence the larger is 32, and the smaller 18.

Ex. (25.) Let $y =$ the man's age, and $x =$ his wife's.

1. Then, by conditions, $x : y :: 3 : 4.$
2. Multiplying extremes, &c., $4x=3y.$
3. By conditions, $x+12 : y+12 :: 5 : 6.$
4. Multiplying extremes, &c., $6x+72=5y+60.$
5. Multiplying (4) by 2, $12x+144=10y+120.$
6. Multiplying (2) by 3, $12x=9y.$
7. Subtracting (6) from (5), $144= y+120.$
8. Transposing, &c., $y=24.$
9. Substituting for y its value in (2), $4x=72.$
10. Dividing, $x=18.$

Hence the man's age is 24, his wife's 18 years.

Ex. (26.) Let $x =$ the days he labored, and $y =$ the days he was absent.

1. By conditions, $x+y= 10.$
2. " " $12x-8y= 40.$
3. Multiplying (1) by 12, $12x+12y=120.$
4. Subtracting (2) from (3), $20y= 80.$
5. Dividing, $y= 4.$
6. Substituting for y its value in (1), $x+4= 10.$
7. Transposing, &c., $x= 6.$

Hence he labored 6 days, and was absent 4 days.

Ex. (27.) Let $x =$ the price of the chaise, and $y =$ the price of the horse.

1. Then, by conditions, $x+y= 208.$
2. " " " $\frac{4x}{7}= \frac{2y}{3}.$
3. Clearing of fractions, $12x= 14y.$
4. Multiplying (1) by 12, $12x+12y=2496.$
5. Subtracting (3) from (4), $26y=2496.$
6. Dividing, $y= 96.$
7. Substituting for y its value in (1), $x+96= 208.$
8. Transposing, &c., $x= 112.$

Hence the chaise cost $112, and the horse $96.

Ex. (28.) Let $2x$ = the sum A lost, and x = the sum B lost.

1. Then, by conditions, $\frac{240-2x}{3}=96-x.$
2. Clearing of fractions, $240-2x=288-3x$
3. Transposing, &c., $x=48.$
4. Multiplying, $2x=96.$

Hence A lost \$96, and B lost \$48.

Ex. (29.) Let x = the time A would finish the work.

As A and B would complete the work in 6 days, having already labored 4 days, it is evident that in one day they would do $\frac{1}{6}$ of the work.

And as B could finish it in 16 days, he would in one day do $\frac{1}{16}$ of the remaining labor.

Therefore, A would do $\frac{1}{6}-\frac{1}{16}=\frac{5}{48}$ of it in one day.

And $\frac{5}{48}$ work : 1 work : : 1 day : x days.

Multiplying extremes, &c., $\frac{5x}{48}=1.$

Dividing, &c., $x=9\frac{3}{5}$ days.

Hence A would finish the work in $9\frac{3}{5}$ days.

Ex. (30.) Let x = the first kind of grain, $40-x$ = the other kind.

1. Then, by conditions, $60\times x+90(40-x)=40\times 80.$
2. Reducing terms, $60x+3600-90x=3200.$
3. And $30x=400.$
4. Dividing, $x=13\frac{1}{3}.$
5. And $40-13\frac{1}{3}=26\frac{2}{3}.$

Hence the first kind of grain was $13\frac{1}{3}$ bushels, and the second kind $26\frac{2}{3}$ bushels.

Ex. (31.) Let x = bushels of corn, and $170-x$ = bushels of barley.

1. Then, $30\times 30+70\times x+90(170-x)=200\times 80.$
2. Reducing, $900+70x+15300-90x=16000.$
3. Collecting terms, $20x=200.$
4. And $x=10.$

5. Substituting for x its value in the conditions, $170-10=160.$

Hence, 10 bushels of corn, and 160 of barley.

Ex. (32.) Let $x =$ the oxen, and $y =$ the cows.

1. By conditions, $\frac{x-6}{2}=y-8.$
2. " " $x-6-10=y-8+2.$
3. Reducing, $x-16=y-6.$
4. Twice the (1), $x-6=2y-16.$
5. Subtracting (3) from (4), $10=y-10.$
6. Transposing, $y=20$, cows.
7. Substituting for y its value in (3), $x-16=20-6.$
8. Transposing, &c., $x=30$, oxen.

Hence there were 30 oxen and 20 cows.

Ex. (33.) Let $x =$ the larger number, and $y =$ the less.

1. By conditions, $x+y=15.$
2. " " $4x=6y.$
3. Multiplying (1) by 4, $4x+4y=60.$
4. Subtracting (2) from (3), $10y=60.$
5. Dividing, $y=6.$
6. Substituting for y its value in (1), $x+6=15.$
7. Transposing, $x=9.$

Hence the numbers are 9 and 6.

Ex. (34.) Let the times A, B and C would perform the work respectively be $= x, y, z$; and let $w =$ the time they would all do it. Then, as A and B would together perform the work in 6 days, it is evident they would in one day perform $\frac{1}{6}$ of the labor; and, for the same reason, A and C would perform in one day $\frac{1}{8}$ of the work, and B and C, in 12 days, $\frac{1}{12}$ of it.

Therefore, $A+B=\frac{1}{6}.$

And $A+C=\frac{1}{8}.$

And $B+C=\frac{1}{12}.$

Therefore, $2A+2B+2C=\frac{3}{8}.$

And $A+B+C=\frac{3}{16}.$

Then, as B and C would do $\frac{1}{12}$ of the work in one day, and A, B and C $\frac{3}{16}$ of it, it is evident that A in one day would do $\frac{3}{16}-\frac{1}{12}=\frac{5}{48}$ of it.

Therefore, $\frac{5}{48}$ work : 1 work : : 1 day : x days.

Multiplying extremes, &c., $\frac{5x}{48}=1.$

Clearing of fractions, $5x=48.$

Dividing, $x=9\frac{3}{5}$ days.

B in one day would do $\frac{3}{16}-\frac{1}{8}=\frac{1}{16}$ of it.

Therefore, $\frac{1}{16}$ work : 1 work : : 1 day : y days.

Multiplying extremes, &c., $\frac{y}{16}=1.$

Clearing of fractions, $y=16$ days.

C in one day would do $\frac{3}{16}-\frac{1}{6}=\frac{1}{48}$ of it.

Therefore, $\frac{1}{48}$ work : 1 work : : 1 day : z days.

Multiplying extremes and means, $\frac{z}{48}=1.$

Clearing of fractions, $z=48$ days.

As A, B and C do $\frac{3}{16}$ of the work in a day,

Therefore, $\frac{3}{16}$ work : 1 work : : 1 day : w days.

Multiplying extremes and means, $\frac{3w}{16}=1.$

Clearing of fractions, $3w=16.$

Dividing, $w=5\frac{1}{3}$ days.

Hence, A would do the work in $9\frac{3}{5}$ days, B in 16 days, C in 48 days; A, B and C together, in $5\frac{1}{3}$ days.

Ex. (35.) Let the four shares be represented respectively by w, x, y, z.

1. Conditions, $w=\frac{x+y+z}{2}.$

2. " $x=\frac{w+y+z}{3}.$

3. " $y=\frac{w+x+z}{4}.$

4. " $w=z+14.$

5. Clearing of fractions, $2w=x+y+z$.
6. " " " $3x=w+y+z$.
7. " " " $4y=w+x+z$.
8. (4.) $w=z+14$.
9. Subtracting (5) from (6), $3w=4x$.
10. Subtracting (8) from (7), $4y=x+2w-14$.
11. Subtracting (4) from (5), $w=x+y-14$.
12. Multiplying (11) by 4, and transposing, $4x+4y=56+4w$.
13. Transposing (10), $-x+4y=-14+2w$.
14. Subtracting (13) from (12), $5x=70+2w$.
15. Multiplying (14) by 4, $20x=280+8w$.
16. Multiplying (9) by 5, $20x=15w$.
17. Subtracting (16) from (15), $=280-7w$.
18. Transposing, $7w=280$.
19. And $w=40$.
20. Substituting for w its value in (9), $4x=120$.
21. Dividing, $x=30$.
22. Substituting for w and x their value in (11), $40=30+y-14$.
23. Transposing, &c., $y=24$.
24. Substituting for w its value in (8), $40=z+14$.
25. Transposing, &c., $z=26$.
26. Adding the values of w, x, y, z, $40+30+24+26=120$.

Hence the whole sum is $120; oldest son's share, $40; second son's, $30; third son's, $24; youngest son's, $26.

NOTE. — In solving the foregoing problems, the pupil should perform each one by eliminating the unknown terms not only by addition and subtraction, but also by comparison and substitution. By so doing he will obtain more knowledge of solving equations than by any other means.

NEGATIVE QUANTITIES.

Art. 153. (p. 106.)

Ex. (8.) Let $\frac{x}{y}$ = the fraction.

1. Then, by conditions, $\frac{x+2}{y}=\frac{1}{4}$.
2. And, " " $\frac{x}{y+2}=\frac{1}{2}$.
3. Clearing of fractions, $4x+8=y$.
4. " " " $2x=y+2$.
5. Subtracting (4) from (3), $2x+8=-2$.
6. Transposing, $2x=-10$.
7. Dividing, $x=-5$.
8. Substituting for x its value in (3), $-20+8=y$.
9. Transposing, $y=-12$.
10. Hence the fraction is $\frac{-5}{-12}$.

Ex. (9.) Let $\frac{x}{y}$ = the fraction.

1. Then, by conditions, $\frac{x+7}{y}=0$.
2. And " " $\frac{x}{y+2}=\frac{1}{0}$.
3. Clearing of fractions, $x+7=0$.
4. " " " $0=y+2$.
5. Transposing (3), $x=-7$.
6. Transposing (4), $y=-2$.
7. Hence the fraction is $\frac{-7}{-2}$.

Ex. (10.) Let $\frac{x}{y}$ = the fraction.

1. Then, by conditions, $\frac{x+4}{y}=0$.
2. And " " $\frac{x}{y-10}=1$.

3. Clearing of fractions, $x+4=0.$
4. " " " $x=y-10.$
5. Transposing (3), $x=-4.$
6. Substituting for x in the (4) its value in the (5), $-4=y-10.$
7. Transposing, $y=6.$
8. Hence the fraction is $\frac{-4}{6}$.

THEOREMS.

Art. 157. (p. 114.)

Ex. (4.) *Ans.* $9a^2b^2+6abm+m^2.$
(5.) *Ans.* $25y^2+40xy+16x^2.$
(6.) *Ans.* $4m^2+12mn+9n^2.$
(7.) *Ans.* $49d^2+28de+4e^2.$
(8.) *Ans.* $4n^2+12nw+9w^2.$

Ex. (9.) *Ans.* $25a^4+20a^2b+4b^2.$
(10.) *Ans.* $1+\frac{1}{2}+\frac{1}{16}.$
(11.) *Ans.* $9+\frac{3}{4}+\frac{1}{64}.$
(12.) *Ans.* $4+2+\frac{1}{4}.$

Art. 159. (p. 115.)

Ex. (2.) *Ans.* $9a^2-12ab+4b^2.$
(3.) *Ans.* $25m^2-10mn+n^2.$
(4.) *Ans.* $16a^2b^2-8abx+x^2.$

Ex. (5.) *Ans.* $9a^4-6a^2b^3+b^6.$
(6.) *Ans.* $x^8-2x^4y^2+y^4.$

EVOLUTION.

Art. 183. (p. 129.)

Ex. (6.)

$$\begin{array}{rl}
 & x^4-2x^3+3x^2-2x+1\ (\ x^2-x+1. \\
 & x^4 \\
\hline
2x^2-x\ ) & -2x^3+3x^2 \\
 & -2x^3+\ x^2 \\
\hline
2x^2-2x+1\ ) & 2x^2-2x+1 \\
 & 2x^2-2x+1 \\
\hline
\end{array}$$

Ex. (7.) $x^6-2x^5+x^4+2x^3-2x^2+1\,(\,x^3-x^2+1.$

$$\begin{array}{rl}
 & x^6 \\ \hline
2x^3-x^2\,) & -2x^5+x^4 \\
 & -2x^5+x^4 \\ \hline
2x^3-2x^2+1\,) & 2x^3-2x^2+1 \\
 & \underline{2x^3-2x^2+1}
\end{array}$$

Ex. (8.) $a^4+4a^3b+10a^2b^2+12ab^3+9b^4\,(\,a^2+2ab+3b^2.$

$$\begin{array}{rl}
 & a^4 \\ \hline
2a^2+2ab\,) & 4a^3b+10a^2b^2 \\
 & 4a^3b+\ \ 4a^2b^2 \\ \hline
2a^2+4ab+3b^2\,) & 6a^2b^2+12ab^3+9b^4 \\
 & \underline{6a^2b^2+12ab^3+9b^4}
\end{array}$$

Ex. (9.) $a^4-2a^3+2a^2-a+\frac{1}{4}\,(\,a^2-a+\frac{1}{2}.$

$$\begin{array}{rl}
 & a^4 \\ \hline
2a^2-a\,) & -2a^3+2a^2 \\
 & -2a^3+\ \ a^2 \\ \hline
2a^2-2a+\frac{1}{2}\,) & a^2-a+\frac{1}{4} \\
 & \underline{a^2-a+\frac{1}{4}}
\end{array}$$

Ex. (10.) $4a^2x^4-12a^3x^3+13a^4x^2-6a^5x+a^6\,(\,2ax^2-3a^2x+a^3.$

$$\begin{array}{rl}
 & 4a^2x^4 \\ \hline
4ax^2-3a^2x\,) & -12a^3x^3+13a^4x^2 \\
 & -12a^3x^3+\ \ 9a^4x^2 \\ \hline
4ax^2-6a^2x+a^3\,) & 4a^4x^2-6a^5x+a^6 \\
 & \underline{4a^4x^2-6a^5x+a^6}
\end{array}$$

Ex. (11.) $\frac{a^2}{b^2}-\frac{4ab}{3bc}+\frac{4b^2}{9c^2}\Big(\frac{a}{b}-\frac{2b}{3c}.$

$$\begin{array}{rl}
 & \frac{a^2}{b^2} \\ \hline
\frac{2a}{b}-\frac{2b}{3c}\Big) & -\frac{4ab}{3bc}+\frac{4b^2}{9c^2} \\
 & \underline{-\frac{4ab}{3bc}+\frac{4b^2}{9c^2}}
\end{array}$$

BY DETACHED COEFFICIENTS.

ART. 183. (p. 130.)

Ex. (4.) $4-16+24-16+4\ (2-4+2.$

4 Hence $2a^2-4a+2$. *Ans.*

$4-4\ )\ -16+24$

$-16+16$

$4-8+2\ )\ 8-16+4$

$8-16+4$

Ex. (5.) $4+0+0+0-12-12+0+0+9+18+9$

4 $(2+0+0+0-3-3.$

$4+0+0+0-3\ )\ +0+0+0-12-12+0+0+9$

$+0+0+0-12-\ 0+0+0+9$

$4+0+0+0-6-3\ )\ -12+0+0+0+18+9$

$-12+0+0+0+18+9$

Hence $2+0+0+0-3-3=2x^5+0x^4+0x^3+0x^2-3x-3=2x^5-3x-3$. *Ans.*

Ex. (6.) $16+24+89+60+100\ (4+3+10=4x^2+3x+10$. *Ans.*

16

$8+3\ )\ 24+89$

$24+\ 9$

$8+6+10\ )\ 80+60+100$

$80+60+100$

Ex. (7.) $9-12+10-28+17-8+16\ (3-2+1-4=3x^3-2x^2+x-4$. *Ans.*

9

$6-2\ )\ -12+10$

$-12+\ 4$

$6-4+1\ )\ 6-28+17$

$6-\ 4+\ 1$

$6-4+2-4\ )\ -24+16-8+16$

$-24+16-8+16$

Ex. (8.) $1+2-1-\frac{2}{1}+\frac{1}{1^2}\Big(1+1-\frac{1}{1}=m+1-\frac{1}{m}$. *Ans.*

$$\begin{array}{r}
1\\
\hline
2+1\ )\ 2-1\\
2+1\\
\hline
2+2-\frac{1}{1}\ )\ -2-\frac{2}{1}+\frac{1}{1^2}\\
-2-\frac{2}{1}+\frac{1}{1^2}\\
\hline
\end{array}$$

CUBE ROOT.

ART. 186. (p. 134.)

Ex. (4.) $x^3+9x^2+27x+27\ (\ x+3$

$$\begin{array}{r}
x^3\\
\hline
3x^2+9x+9\ )\ 9x^2+27x+27\\
9x^2+27x+27\\
\hline
\end{array}$$

The divisor is obtained in the following manner.

$$3(x)^2+3(3\times x)+(3)^2=3x^2+9x+9.$$

Ex. (5.) $1-6y+12y^2-8y^3\ (\ 1-2y.$

$$\begin{array}{r}
1\\
\hline
3-6y+4y^2\ )\ -6y+12y^2-8y^3\\
-6y+12y^2-8y^3\\
\hline
\end{array}$$

The divisor, $3(1)^2+3(1\times-2y)+(-2y)^2=3-6y+4y^2$.

Ex. (6.) $a^6-6a^5+40a^3-96a-64\ (\ a^2-2a-4.$

$$\begin{array}{r}
a^6\\
\hline
3a^4-6a^3+4a^2\ )\ -6a^5+40a^3+96a\\
.\quad -6a^5+12a^4-\ 8a^3\\
\hline
3a^4-12a^3+24a+16\ )\ -12a^4+48x^3-96a-64\\
-12a^4+48x^3-96a-64\\
\hline
\end{array}$$

First divisor, $3(a^2)+3(a^2\times-2a)+(-2a)^2=3a^4+6a^3+4a^2$.

Second divisor, $3(a^2-2a)^2+(3(a^2-2a)-4)+(-4)^2=3a^4-12a^3+24a+16$.

Ex. (7.)

$$\begin{array}{rl}
 & a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\
 & a^3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\,a+b+c. \\
\hline
3a^2+3ab+b^2\,) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
3a^2+6ab+3b^2+3ac+3bc+c^2) & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\
 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\
\hline
\end{array}$$

First divisor, $3(a)^2+3(a\times b)+(b)^2=3a^2+3ab+b^2$.

Second divisor, $3(a+b)^2+3(a+b)c+(c)^2=3a^2+6ab+3b^2+3ac+3bc+c^2$.

BY DETACHED COEFFICIENTS. (p. 135.)

Ex. (3.)

$$\begin{array}{rl}
 & 1+8+24+32+16\,(\,1+4+4=x^2+4x+4. \\
 & 1 \\
\hline
2+4\,) & 8+24 \\
 & 8+16 \\
\hline
2+8+4\,) & 8+32+16 \\
 & 8+32+16 \\
\hline
\end{array}$$

$$\begin{array}{rl}
 & 1+4+4\,(\,1+2=x+2.\quad \textit{Ans.} \\
 & 1 \\
\hline
2+2\,) & 4+4 \\
 & 4+4 \\
\hline
\end{array}$$

In this question we extract the square root of the square root.

Ex. (4.)

$$\begin{array}{rll}
 & 1+0+0+3+0+0+3-1\,(\,1+0+0-1= & \\
 & 1 & x^3+0x^2+0x-y= \\
\hline
3+0+0+3+1\,) & 0+0+3+0+0+3-1 & x^3-y. \\
 & 0+0+3+0+0+3-1 & \\
\end{array}$$

ART. **188.** (p. 138.)

Ex. (4.)

$$\begin{array}{rl}
 & m^6-6m^5+40m^3-96m-64\,(\,m^2-2m-4. \\
 & m^6 \\
\hline
3m^4\,) & -6m^5 \\
\hline
 & m^6-6m^5+12m^4-8m^3 \\
\hline
3m^4\,) & -12m^4-48m^3-96m-64 \\
\hline
 & m^6-6m^5+40m^3-96m-64 \\
\end{array}$$

Ex. (5.) $32x^5-80x^4+80x^3-40x^2+10x-1\ (\,2x-1.$

$$\begin{array}{l} \quad 32x^5 \\ \hline 80x^4\,)\ -80x^4 \\ \hline \quad 32x^5-80x^4+80x^3-40x^2+10x-1. \end{array}$$

SURDS, OR RADICAL QUANTITIES.

Art. 191. (p. 139.)

Ex. (4.) $3a^2\times 3a^2=9a^4$; whence $\sqrt{9a^4}$.

(5.) $\frac{x^2}{3}\times\frac{x^2}{3}\times\frac{x^2}{3}=\frac{x^6}{27}$; whence $\sqrt[3]{\frac{x^6}{27}}$.

(6.) $x^3\times x^3\times x^3\times x^3\times x^3=x^{15}$; whence $\sqrt[5]{x^{15}}$.

(7.) $\frac{x^2}{x-y}\times\frac{x^2}{x-y}\times\frac{x^2}{x-y}\times\frac{x^2}{x-y}=\frac{x^8}{(x-y)^4}$; whence $\left(\frac{x^8}{(x-y)^4}\right)^{\frac{1}{4}}$.

(8.) $(x-y^2)(x-y^2)=(x-y^2)^2$; whence $(\,(x-y^2)^2)^{\frac{1}{2}}$.

Art. 192. (p. 140.)

Ex. (12.) $\frac{1}{3}\sqrt{a}=\sqrt{\frac{1}{3}\times\frac{1}{3}}\times\sqrt{a}=\sqrt{\frac{a}{9}}$.

(13.) $\frac{1}{2}\sqrt[4]{b^2}=\sqrt[4]{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}\times\sqrt[4]{b^2}=\sqrt[4]{\frac{b^2}{16}}$.

(14.) $3^n\times\sqrt[n]{m}=\sqrt[n]{3^n m}$.

(16.) $\sqrt[3]{\frac{2x}{3}\times\frac{2x}{3}\times\frac{2x}{3}}\times\sqrt[3]{\frac{9}{4x^2}}=\sqrt[3]{\frac{72x^3}{108x^2}}=\sqrt[3]{\frac{2x}{3}}$.

Art. 194. (p. 141.)

Ex. (2.) Here $\frac{1}{3}\div\frac{1}{6}=\frac{1}{3}\times\frac{6}{1}=2$, the first index.
And $\frac{1}{2}\div\frac{1}{6}=\frac{1}{2}\times\frac{6}{1}=3$, the second index.

Therefore, $3^{\frac{1}{3}}=\sqrt[6]{3^2}$, or $\sqrt[6]{9}$, and $5^{\frac{1}{2}}=\sqrt[6]{5^3}=\sqrt[6]{125}$.

Ex. (3.) Here $\frac{2}{1} \div \frac{1}{4} = 8$, the first index.
And $\frac{1}{2} \div \frac{1}{4} = 2$, the second index.
Therefore $a^2 = \sqrt[4]{a^8}$, and $a^{\frac{1}{2}} = \sqrt[4]{a^2}$.

Ex. (4.) Here $\frac{1}{2} \div \frac{1}{8} = 4$, the first index.
And $\frac{3}{4} \div \frac{1}{8} = 6$, the second index.
Therefore $3a^{\frac{1}{2}} = 3\sqrt[8]{a^4}$, and $2a^{\frac{3}{4}} = 2\sqrt[8]{a^6}$.

Ex. (5.) Here $\frac{1}{4} \div \frac{1}{12} = 3$.
And $\frac{1}{3} \div \frac{1}{12} = 4$.
Whence $5x^{\frac{1}{4}} = 5\sqrt[12]{x^3}$, and $6y^{\frac{1}{3}} = 6\sqrt[12]{y^4}$.

Art. 195. (p. 142.)

Ex. (3.) $\frac{2}{3}$ and $\frac{1}{2} = \frac{4}{6}$ and $\frac{3}{6}$.
Hence $2^{\frac{2}{3}} = 2^{\frac{4}{6}} = (2^4)^{\frac{1}{6}} = 16^{\frac{1}{6}} = \sqrt[6]{16}$.
And $3^{\frac{1}{2}} = 3^{\frac{3}{6}} = (3^3)^{\frac{1}{6}} = (27)^{\frac{1}{6}} = \sqrt[6]{27}$.

Ex. (4.) $\frac{1}{2}$ and $\frac{1}{4} = \frac{2}{4}$ and $\frac{1}{4}$.
Hence $a^{\frac{1}{2}} = a^{\frac{2}{4}} = \sqrt[4]{a^2}$.
And $b^{\frac{1}{4}} = (b)^{\frac{1}{4}} = \sqrt[4]{b}$.

Ex. (5.) $\frac{1}{m}$ and $\frac{1}{n} = \frac{n}{mn}$ and $\frac{m}{mn}$.
Therefore $x^{\frac{1}{m}} = (x^n)^{\frac{1}{mn}} = \sqrt[mn]{x^n}$.
And $y^{\frac{1}{n}} = (y^m)^{\frac{1}{mn}} = \sqrt[mn]{y^m}$.

Art. 196. (p. 143.)

Ex. (3.) $\sqrt{75} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$.
(4.) $\sqrt[4]{80} = \sqrt[4]{16} \times \sqrt[4]{5} = 2\sqrt[4]{5}$.
(6.) $\sqrt[3]{54a^5x^4} = \sqrt[3]{27a^3x^3} \times \sqrt[3]{2a^2x} = 3ax\sqrt[3]{2a^2x}$.

Art. 198. (p. 144.)

Ex. (3.) $2\sqrt[3]{40} = \sqrt[3]{2 \times 2 \times 2 \times 40} = \sqrt[3]{320} = \sqrt[3]{64 \times 5} = 4\sqrt[3]{5}$

ART. 199. (p. 144.)

Ex. (3.) $\sqrt{\frac{2}{7}}=\sqrt{\frac{2}{7}\times\frac{7}{7}}=\sqrt{\frac{14}{49}}=\sqrt{\frac{1}{49}\times\frac{14}{1}}=\frac{1}{7}\sqrt{14}.$

(4.) $\sqrt[3]{\frac{2}{5}}=\sqrt[3]{\frac{2}{5}\times\frac{25}{25}}=\sqrt[3]{\frac{50}{125}}=\sqrt[3]{\frac{1}{125}\times\frac{50}{1}}=\frac{1}{5}\sqrt[3]{50}.$

(5.) $\sqrt[4]{\frac{1}{2}}=\sqrt[4]{\frac{1}{2}\times\frac{8}{8}}=\sqrt[4]{\frac{8}{16}}=\sqrt[4]{\frac{1}{16}\times\frac{8}{1}}=\frac{1}{2}\sqrt[4]{8}.$

EXAMPLES TO EXERCISE THE FOREGOING RULES.

Ex. (1.) $\sqrt{125}=\sqrt{25\times5}=5\sqrt{5}.$

(2.) $\sqrt{80a^2x^3}=\sqrt{16a^2x^2}\times\sqrt{5x}=4ax\sqrt{5x}.$

(3.) $\sqrt[3]{189a^4b^3c^2}=\sqrt[3]{27a^3b^3}\times\sqrt[3]{7ac^2}=3ab\sqrt[3]{7ac^2}.$

(4.) $7\sqrt{80}=\sqrt{3920}=\sqrt{784}\times\sqrt{5}=28\sqrt{5}.$

(5.) $\frac{3}{7}\sqrt{\frac{4}{5}}=\sqrt{\frac{36}{245}}=\sqrt{\frac{36}{245}\times\frac{5}{5}}=\sqrt{\frac{180}{1225}}=\sqrt{\frac{36}{1225}\times\frac{5}{1}}$
$=\frac{6}{35}\sqrt{5}.$

(6.) $\frac{3}{11}\sqrt{\frac{4}{7}}=\sqrt{\frac{36}{847}\times\frac{7}{7}}=\sqrt{\frac{252}{5929}}=\sqrt{\frac{36}{5929}\times\frac{7}{1}}=\frac{6}{77}\sqrt{7}.$

(7.) $\sqrt{96a^2x^3}=\sqrt{16a^2x^2}\times\sqrt{6x}=4ax\sqrt{6x}.$

(8.) $\frac{3}{7}\sqrt[3]{56x^4+64y^3}=\frac{3}{7}\sqrt[3]{8\times(7x^4+8y^3)}=\frac{6}{7}\sqrt[3]{7x^4+8y^3}.$

ADDITION OF SURD QUANTITIES.

ART. 200. (p. 146.)

Ex. (7.) Required the sum of $\sqrt{27}$ and $\sqrt{48}$.

First $\sqrt{27}=\sqrt{9\times3}=3\sqrt{3}.$

And $\sqrt{48}=\sqrt{16\times3}=4\sqrt{3}.$

Then $3\sqrt{3}+4\sqrt{3}=7\sqrt{3}.$

Ex. (8.) Required the sum of $\sqrt{50}$ and $\sqrt{72}$.

First $\sqrt{50}=\sqrt{25\times2}=5\sqrt{2}.$

And $\sqrt{72}=\sqrt{36\times2}=6\sqrt{2}.$

Then $5\sqrt{2}+6\sqrt{2}=11\sqrt{2}.$

Ex. (9.) Find the sum of $\sqrt{180}$ and $\sqrt{405}$.

First $\sqrt{180}=\sqrt{36\times5}=6\sqrt{5}$.

And $\sqrt{405}=\sqrt{81\times5}=9\sqrt{5}$.

Then $6\sqrt{5}+9\sqrt{5}=15\sqrt{5}$.

Ex. (10.) Find the sum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$.

First $\sqrt[3]{40}=\sqrt[3]{8\times5}=2\sqrt[3]{5}$.

And $\sqrt[3]{135}=\sqrt[3]{27\times5}=3\sqrt[3]{5}$.

Then $2\sqrt[3]{5}+3\sqrt[3]{5}=5\sqrt[3]{5}$.

Ex. (11.) Find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$.

First

$4\sqrt[3]{54}=\sqrt[3]{4\times4\times4\times54}=\sqrt[3]{3456}=\sqrt[3]{1728\times2}=12\sqrt[3]{2}$.

And

$5\sqrt[3]{128}=\sqrt[3]{5\times5\times5\times128}=\sqrt[3]{16000}=\sqrt[3]{8000\times2}=20\sqrt[3]{2}$.

Then $12\sqrt[3]{2}+20\sqrt[3]{2}=32\sqrt[3]{2}$.

Ex. (12.) Find the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$.

First $\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{1}{4}\times\frac{2}{2}}=\sqrt[3]{\frac{2}{8}}=\sqrt[3]{\frac{1}{8}\times\frac{2}{1}}=\frac{1}{2}\sqrt[3]{2}$.

And $\sqrt[3]{\frac{1}{32}}=\sqrt[3]{\frac{1}{32}\times\frac{2}{2}}=\sqrt[3]{\frac{2}{64}}=\sqrt[3]{\frac{1}{64}\times\frac{2}{1}}=\frac{1}{4}\sqrt[3]{2}$.

Then $\frac{1}{2}\sqrt[3]{2}+\frac{1}{4}\sqrt[3]{2}=\frac{3}{4}\sqrt[3]{2}$.

Ex. (13.) Required the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$.

First $3\sqrt{a^2b}=3a\sqrt{b}$.

And $5\sqrt{16a^4b}=20a^2\sqrt{b}$.

Then $3a\sqrt{b}+20a^2\sqrt{b}=(3a+20a^2)\sqrt{b}$.

SUBTRACTION OF SURD QUANTITIES.

ART. 201. (p. 146.)

Ex. (3.) Required the difference between $2\sqrt{50}$ and $\sqrt{18}$.

First $2\sqrt{50}=\sqrt{2\times2\times50}=\sqrt{200}=\sqrt{100\times2}=10\sqrt{2}$.

And $\sqrt{18}=\sqrt{9\times2}=3\sqrt{2}$.

Then $10\sqrt{2}-3\sqrt{2}=7\sqrt{2}$.

Ex. (4.) What is the difference between $2\sqrt[3]{320}$ and $3\sqrt[3]{40}$?

First

$2\sqrt[3]{320}=\sqrt[3]{2\times2\times2\times320}=\sqrt[3]{2560}=\sqrt[3]{512\times5}=8\sqrt[3]{5}.$

And $3\sqrt[3]{40}=\sqrt[3]{3\times3\times3\times40}=\sqrt[3]{1080}=\sqrt[3]{216\times5}=6\sqrt[3]{5}.$

Then $8\sqrt[3]{5}-6\sqrt[3]{5}=2\sqrt[3]{5}.$

Ex. (5.) Required the difference between $\sqrt{75}$ and $\sqrt{48}.$

First $\sqrt{75}=\sqrt{25\times3}=5\sqrt{3}.$

And $\sqrt{48}=\sqrt{16\times3}=4\sqrt{3}.$

Then $5\sqrt{3}-4\sqrt{3}=\sqrt{3}.$

Ex. (6.) Required the difference of $\sqrt[3]{256}$ and $\sqrt[3]{32}.$

First $\sqrt[3]{256}=\sqrt[3]{64\times4}=4\sqrt[3]{4}.$

And $\sqrt[3]{32}=\sqrt[3]{8\times4}=2\sqrt[3]{4}.$

Then $4\sqrt[3]{4}-2\sqrt[3]{4}=2\sqrt[3]{4}.$

Ex. (7.) Required the difference of $\sqrt[3]{\frac{3}{4}}$ and $\sqrt[3]{\frac{2}{9}}.$

First $\sqrt[3]{\frac{3}{4}}=\sqrt[3]{\frac{3}{4}\times\frac{2}{2}}=\sqrt[3]{\frac{6}{8}}=\sqrt[3]{\frac{1}{8}\times\frac{6}{1}}=\frac{1}{2}\sqrt[3]{6}.$

And $\sqrt[3]{\frac{2}{9}}=\sqrt[3]{\frac{2}{9}\times\frac{3}{3}}=\sqrt[3]{\frac{6}{27}}=\sqrt[3]{\frac{1}{27}\times\frac{6}{1}}=\frac{1}{3}\sqrt[3]{6}.$

Then $\frac{1}{2}\sqrt[3]{6}-\frac{1}{3}\sqrt[3]{6}=\frac{1}{6}\sqrt[3]{6}.$

Ex. (8.) Required the difference of $\sqrt[3]{\frac{3}{5}}$ and $\sqrt[3]{\frac{25}{9}}.$

First $\sqrt[3]{\frac{3}{5}}=\sqrt[3]{\frac{3}{5}\times\frac{25}{25}}=\sqrt[3]{\frac{75}{125}}=\sqrt[3]{\frac{1}{125}\times\frac{75}{1}}=\frac{1}{5}\sqrt{75}.$

And $\sqrt[3]{\frac{25}{9}}=\sqrt[3]{\frac{25}{9}\times\frac{3}{3}}=\sqrt[3]{\frac{75}{27}}=\sqrt[3]{\frac{1}{27}\times\frac{75}{1}}=\frac{1}{3}\sqrt{75}.$

Then $\frac{1}{3}\sqrt[3]{75}-\frac{1}{5}\sqrt[3]{75}=\frac{2}{15}\sqrt{75}.$

Ex. (9.) Find the difference of $\frac{3}{7}\sqrt[3]{a^3b}$ and $\frac{2}{5}\sqrt[3]{a^6b}.$

First $\frac{3}{7}\sqrt[3]{a^3b}=\frac{3a}{7}\sqrt[3]{b}.$

And $\frac{2}{5}\sqrt[3]{a^6b}=\frac{2a^2}{5}\sqrt[3]{b}.$

Then $\frac{3a}{7}\sqrt[3]{b}-\frac{2a^2}{5}\sqrt[3]{b}=\left(\frac{15a}{35}-\frac{14a^2}{35}\right)\sqrt[3]{b}.$

Ex. (10.) From $\sqrt{4ax^2}$ take $3x\sqrt{9a}$.

First $\sqrt{4ax^2}=\sqrt{4x^2\times a}=2x\sqrt{a}.$

And $3x\sqrt{9a}=\sqrt{3x\times 3x\times 9a}=\sqrt{81x^2a}=9x\sqrt{a}.$

Then $2x\sqrt{a}-9x\sqrt{a}=-7x\sqrt{a}.$

MULTIPLICATION OF SURDS.

Art. 204. (p. 148.)

Ex. (5.) Multiply $4\sqrt{12}$

By $3\sqrt{2}$

$12\sqrt{24}=12\sqrt{4\times 6}=24\sqrt{6}.$

Ex. (6.) Multiply $3\sqrt{2}$

By $2\sqrt{8}$

$6\sqrt{16}=24.$

Ex. (7.) Multiply $\frac{1}{3}\sqrt[3]{4}$

By $\frac{3}{4}\sqrt[3]{12}$

$\frac{1}{4}\sqrt[3]{48}=\frac{1}{4}\sqrt[3]{8\times 6}=\frac{1}{2}\sqrt[3]{6}.$

Ex. (8.) Multiply $\frac{5}{3}\sqrt{\frac{3}{8}}$

By $\frac{9}{10}\sqrt{\frac{3}{5}}$

$\frac{3}{2}\sqrt{\frac{9}{40}}=\frac{3}{2}\sqrt{\frac{9}{4}\times\frac{1}{10}}=\frac{3}{2}\times\frac{3}{2}\sqrt{\frac{1}{10}}=\frac{9}{4}\sqrt{\frac{1}{10}}.$

Ex. (9.) Multiply $7\sqrt[3]{18}$

By $5\sqrt[3]{4}$

$35\sqrt[3]{72}=35\sqrt[3]{8\times 9}=70\sqrt[3]{9}.$

Ex. (10.) Multiply $\frac{1}{4}\sqrt[3]{6}$

By $\frac{2}{15}\sqrt[3]{17}$

$\frac{1}{30}\sqrt[3]{102}.$

Ex. (11.) Multiply $2a^{\frac{2}{3}}$

By $a^{\frac{4}{3}}$

$2a^2.$

Ex. (12.) Multiply $(a+b)^{\frac{1}{3}}$

By $(a+b)^{\frac{3}{4}}$.

$$(a+b)^{\frac{1}{3}}=(a+b)^{\frac{4}{12}}$$
$$(a+b)^{\frac{3}{4}}=(a+b)^{\frac{9}{12}}$$
$$\overline{\phantom{(a+b)^{\frac{1}{3}}=(a+b)^{\frac{4}{12}}}}$$
$$(a+b)^{\frac{13}{12}}=\sqrt[12]{(a+b)^{13}}.$$

Ex. (15.) Multiply $\sqrt{a}+\sqrt{b}+\sqrt{c}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

$$\begin{array}{llll}
\sqrt{a}+\sqrt{b}+\sqrt{c} & & & \\
\sqrt{a}+\sqrt{b}-\sqrt{c} & & & \\
\hline
a+\sqrt{ab}+\sqrt{ac} & & & \\
\quad\sqrt{ab}+ & b+\sqrt{bc} & & \\
 & -\sqrt{ac}-\sqrt{bc}-c & & \\
\hline
a & +b & -c+2\sqrt{ab}. &
\end{array}$$

Note.—Similar quantities are multiplied by adding their exponents. See Art. 82.

DIVISION OF SURD QUANTITIES.

Art. 205. (p. 150.)

Ex. (5.) $\dfrac{6\sqrt{54}}{3\sqrt{2}}=2\sqrt{27}=2\sqrt{9}\times3=6\sqrt{3}.$

Ex. (6.) $\dfrac{4\sqrt[3]{72}}{2\sqrt[3]{18}}=2\sqrt[3]{4}.$

(7.) $\dfrac{4\sqrt{50}}{2\sqrt{5}}=2\sqrt{10}.$

Ex. (8.) $\dfrac{6\sqrt[3]{100}}{3\sqrt[3]{5}}=2\sqrt[3]{20}.$

(9.) $\dfrac{\sqrt{20}+\sqrt{12}}{\sqrt{5}+\sqrt{3}}=\sqrt{4}=2.$

Ex. (10.) $\dfrac{32\frac{2}{5}\sqrt{a}}{13\frac{3}{4}\sqrt[3]{b}}=\dfrac{\frac{162}{5}a^{\frac{3}{6}}}{\frac{55}{4}b^{\frac{2}{6}}}=\frac{648}{275}\left(\dfrac{a^3}{b^2}\right)^{\frac{1}{6}}.$

TO INVOLVE A SURD QUANTITY.

Art. 207. (p. 152.)

Ex. (3.) Required the square of $3\sqrt[3]{3}$.

$$(3\sqrt[3]{3})^2=9\sqrt[3]{9}$$

Ex. (4.) Required the cube of $17\sqrt{21}$.

$$(17\sqrt{21})^3=4913\sqrt{9261}=4913\sqrt{441\times21}=103173\sqrt{21}.$$

Ex. (5.) What is the fourth power of $\frac{1}{6}\sqrt{6}$?

$$(\tfrac{1}{6}\sqrt{6})^4=\tfrac{1}{1296}\sqrt{1296}=\tfrac{36}{1296}=\tfrac{1}{36}.$$

Ex. (6.) Required the cube of $\sqrt{3}$.

$$(\sqrt{3})^3=\sqrt{27}=\sqrt{9\times3}=3\sqrt{3}.$$

Ex. (7.) Required the third power of $\frac{1}{3}\sqrt{3}$.

$$(\tfrac{1}{3}\sqrt{3})^3=\tfrac{1}{27}\sqrt{27}=\tfrac{1}{27}\sqrt{9\times3}=\tfrac{1}{9}\sqrt{3}.$$

Ex. (8.) Required the fourth power of $\frac{1}{2}\sqrt{2}$.

$$(\tfrac{1}{2}\sqrt{2})^4=\tfrac{1}{16}\sqrt{16}=\tfrac{4}{16}=\tfrac{1}{4}.$$

Ex. (9.) What is the *m*th power of $a^{\frac{1}{n}}$?

$$(a^{\frac{1}{n}})^m=a^{\frac{1}{n}\times m}=a^{\frac{m}{n}}.$$

Ex. (10.) Required the square of $2+\sqrt{3}$.

$$(2+\sqrt{3})^2=4+4\sqrt{3}+3=7+4\sqrt{3}.$$

Ex. (11.) What is the $\frac{r}{s}$th power of $a^{\frac{p}{q}}$?

$$(a^{\frac{p}{q}})^{\frac{r}{s}}=a^{\frac{p}{q}\times\frac{r}{s}}=a^{\frac{pr}{qs}}.$$

ART. 210. (p. 156.)

Ex. (1.) Find a multiplier that shall make $\sqrt{5}-\sqrt{2}$ rational.

$$\begin{array}{r}
\sqrt{5}-\sqrt{2} \\
\underline{\sqrt{5}+\sqrt{2}} \\
5-\sqrt{10} \\
\underline{+\sqrt{10}-2} \\
5-2=3.
\end{array}$$

Ex. (2.) Find a multiplier that shall make $\sqrt{7}+\sqrt{6}$ rational.

$$\begin{array}{r}
\sqrt{7}+\sqrt{6} \\
\underline{\sqrt{7}-\sqrt{6}} \\
7+\sqrt{42} \\
\underline{-\sqrt{42}-6} \\
7-6=1.
\end{array}$$

Ex. (3.) Find a multiplier that shall make $\sqrt{10}-\sqrt{2}$ rational.

$$\begin{array}{l}
\sqrt{10}-\sqrt{2}\\
\sqrt{10}+\sqrt{2}\\
\hline
10-\sqrt{20}\\
\quad +\sqrt{20}-2\\
\hline
10-2=8.
\end{array}$$

Ex. (4.) Find a multiplier that shall make $\sqrt{a}+\sqrt{b}+\sqrt{c}$ rational.

$$\begin{array}{l}
\sqrt{a}+\sqrt{b}+\sqrt{c}\\
\sqrt{a}-\sqrt{b}-\sqrt{c}\\
\hline
a+\sqrt{ab}+\sqrt{ac}\\
\quad -\sqrt{ab}-b-\sqrt{bc}\\
\qquad\qquad -\sqrt{ac}-\sqrt{bc}-c\\
\hline
a-b-c-2\sqrt{bc}\\
a-b-c+2\sqrt{bc}\\
\hline
a^2-ab-ac-2a\sqrt{bc}\\
\quad -ab+b^2+bc+2b\sqrt{bc}\\
\quad -ac+bc+c^2+2c\sqrt{bc}\\
\qquad 2a\sqrt{bc}-2b\sqrt{bc}-2c\sqrt{bc}-4bc\\
\hline
a^2-2ab-2ac-2bc+b^2+c^2.
\end{array}$$

Ex. (5.) Find a multiplier that shall make $\sqrt[4]{3}-\sqrt[4]{1}$ rational.

$$\begin{array}{l}
\sqrt[4]{3}-\sqrt[4]{1}\\
\sqrt[4]{3}+\sqrt[4]{1}\\
\hline
\sqrt{3}-\sqrt[4]{3}\\
\quad +\sqrt[4]{3}-\sqrt{1}\\
\hline
\sqrt{3}-\sqrt{1}\\
\sqrt{3}+\sqrt{1}\\
\hline
3-\sqrt{3}\\
\quad +\sqrt{3}-1\\
\hline
3-1=2.
\end{array}$$

Art. 211. (p. 158.)

Ex. (7.) Reduce $\frac{\sqrt{7}}{\sqrt{5}+\sqrt{3}}$ to a fraction that shall have a rational denominator.

$$\frac{\sqrt{7}}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{35}-\sqrt{21}}{5-3}=\frac{\sqrt{35}-\sqrt{21}}{2}.$$

Ex. (8.) Reduce $\frac{\sqrt{3}}{3-\sqrt{1}}$ to an equivalent fraction having a rational denominator.

$$\frac{\sqrt{3}}{3-\sqrt{1}}\times\frac{3+\sqrt{1}}{3+\sqrt{1}}=\frac{3\sqrt{3}+\sqrt{3}}{9-1}=\frac{\sqrt{3}}{2}.$$

Ex. (9.) Reduce the fraction $\frac{5}{\sqrt{5}-\sqrt{2}}$ to an equivalent fraction having a rational denominator.

$$\frac{5}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{5\sqrt{5}+5\sqrt{2}}{5-2}=\frac{5\sqrt{5}+5\sqrt{2}}{3}.$$

Ex. (10.) Reduce the fraction $\frac{10}{\sqrt{5}+\sqrt{3}}$ to an equivalent fraction having a rational denominator.

$$\frac{10}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{10\sqrt{5}-10\sqrt{3}}{5-3}=\frac{5\sqrt{5}-5\sqrt{3}}{1}.$$

Ex. (11.) Reduce $\frac{1}{\sqrt{5}+\sqrt{7}}$ to a fraction having a rational denominator.

$$\frac{1}{\sqrt{5}+\sqrt{7}}\times\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}-\sqrt{7}}=\frac{\sqrt{5}-\sqrt{7}}{5-7}=\frac{\sqrt{5}-\sqrt{7}}{-2}.$$

Ex. (12.) Reduce the fraction $\frac{3}{\sqrt{5}-\sqrt{2}}$ to an equivalent fraction having a rational denominator.

$$\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=\frac{3\sqrt{5}+3\sqrt{2}}{3}=\sqrt{5}+\sqrt{2}.$$

ART. 212. (p. 159.)

Ex. (4.) Let $3-\sqrt{5}$ be reduced to a general surd.

$$(3-\sqrt{5})^2=9-6\sqrt{5}+5=\sqrt{(14-6\sqrt{5})}.$$

Ex. (5.) Let $\sqrt{2}+2\sqrt{6}$ be changed to a general surd.

$$(\sqrt{2}+2\sqrt{6})^2=2+4\sqrt{12}+24=26+4\sqrt{4\times 3}=$$
$$\sqrt{(26-8\sqrt{3})}.$$

Ex. (6.) It is required to change $4-\sqrt{7}$ to a general surd.

$$(4-\sqrt{7})^2=16-8\sqrt{7}+7=\sqrt{(23-8\sqrt{7})}.$$

Ex. (7.) Let $7\sqrt[3]{3}-3\sqrt[3]{9}$ be changed to a general surd.

$$(7\sqrt[3]{3}-3\sqrt[3]{9})^3=1029-243-3\times 3\times 7\sqrt[3]{9}\times 3(7\sqrt[3]{3}-3\sqrt[3]{9})$$
$$=786-63\sqrt[3]{27}(7\sqrt[3]{3}-3\sqrt[3]{9})=786-189(7\sqrt[3]{3}-3\sqrt[3]{9}).$$

Therefore, $7\sqrt[3]{3}-3\sqrt[3]{9}=\sqrt[3]{(786-189(7\sqrt[3]{3}-3\sqrt[3]{9}))}=$
$$\sqrt[3]{(786-1323\sqrt[3]{3}+567\sqrt[3]{9})}.$$

ART. 213. (p. 161.)

Ex. (3.) What is the square root of $6+\sqrt{20}$?

$$\sqrt{\left(\frac{6+\sqrt{6^2-20}}{2}\right)}=\sqrt{5}.$$
$$\sqrt{\left(\frac{6-\sqrt{6^2-20}}{2}\right)}=1.$$

Therefore, $\sqrt{(6+\sqrt{20})}=1+\sqrt{5}.$

Ex. (4.) What is the square root of $6+2\sqrt{5}$?

$$\sqrt{\left(\frac{6+\sqrt{6^2-20}}{2}\right)}=\sqrt{5}.$$
$$\sqrt{\left(\frac{6-\sqrt{6^2-20}}{2}\right)}=1.$$

Therefore, $\sqrt{(6+2\sqrt{5})}=\sqrt{5}+1.$

Ex. (5.) What is the square root of $12+2\sqrt{35}$?

$$\sqrt{\left(\frac{12+\sqrt{12^2-140}}{2}\right)}=\sqrt{7}.$$
$$\sqrt{\left(\frac{12-\sqrt{12^2-140}}{2}\right)}=\sqrt{5}.$$

Therefore, $\sqrt{(12+2\sqrt{35})}=\sqrt{7}+\sqrt{5}$

Ex. (6.) What is the square root of $36\pm10\sqrt{11}$?

$$\sqrt{\left(\frac{36\pm\sqrt{36^2-1100}}{2}\right)}=5.$$

$$\sqrt{\left(\frac{36\mp\sqrt{36^2-1100}}{2}\right)}=\pm\sqrt{11}.$$

Therefore, $\sqrt{(36\pm10\sqrt{11})}=5\pm\sqrt{11}.$

Ex. (7.) What is the square root of $7-2\sqrt{10}$?

$$\sqrt{\left(\frac{7-\sqrt{7^2-40}}{2}\right)}=\sqrt{2}.$$

$$\sqrt{\left(\frac{7+\sqrt{7^2-40}}{2}\right)}=\sqrt{5}.$$

Therefore, $\sqrt{(7-2\sqrt{10})}=\sqrt{5}-\sqrt{2}.$

Ex. (8.) What is the square root of $1+4\sqrt{-3}$?

$$\sqrt{\left(\frac{1+\sqrt{1^2+48}}{2}\right)}=2.$$

$$\sqrt{\left(\frac{1-\sqrt{1^2+48}}{2}\right)}=\pm\sqrt{-3}.$$

Therefore, $\sqrt{(1+4\sqrt{-3})}=2\pm\sqrt{-3}.$

QUADRATIC EQUATIONS.

Art. 217. (p. 165.)

Ex. (5.) Given $7x^2-5=3x^2+11$ to find x.

Conditions, $7x^2-5=3x^2+11.$
Transposing, $7x^2-3x^2=11+5.$
Reducing, $4x^2=16.$
Dividing, $x^2=4.$
Extracting square root, $x=\pm2.$

Ex. (6.) Given $4x^2+15=7x^2-417$ to find x.

Conditions, $4x^2+15=7x^2-417$

Transposing, $4x^2-7x^2=-417-15.$

Reducing, $-3x^2=-432.$

Dividing by -3, $x^2=144.$

Extracting square root, $x=\pm12.$

Ex. (7.) Given $3x^2+7=\frac{5x^2}{4}+35$ to find x.

Conditions, $3x^2+7=\frac{5x^2}{4}+35.$

Clearing of fractions, $12x^2+28=5x^2+140.$

Transposing, $12x^2-5x^2=140-28.$

Reducing, $7x^2=112.$

Dividing, $x^2=16.$

Extracting square root, $x=\pm4.$

Ex. (8.) Given $ax^2+n=m-c$ to find x.

Conditions, $ax^2+n=m-c.$

Transposing, $ax^2=m-c-n.$

Dividing, $x^2=\frac{m-c-n}{a}.$

Extracting square root, $x=\pm\sqrt{\frac{m-c-n}{a}}.$

Ex. (9.) Given $x^2-ab=d$ to find x.

Conditions, $x^2-ab=d.$

Transposing, $x^2=d+ab.$

Extracting square root, $x=\pm\sqrt{d+ab}.$

Ex. (11.) Let x = the length or breadth.

Then, $x^2=$ the contents.

Therefore, $x^2=160\times10=1600.$

And $x=40$ rods.

Ex. (12.) Let $2x$ = B's capital and A's per cent.

Then x = B's gain per cent.

And $100 : 2x :: x : 32.$

Multiplying extremes, $2x^2=3200$.
Dividing, $x^2=1600$.
Evolving, $x=40$.
$2x=80$, B's capital.
$100+80=180 : 100 :: 27 : 15$, A's capital.

Ex. (13.) Let x = the side of the larger field, and $\frac{3x}{5}$ = the side of the smaller.

Then, $x^2-\frac{9x^2}{25}=25600$.
Clearing of fractions, $25x^2-9x^2=640000$.
Uniting terms, $16x^2=640000$.
Dividing, $x^2=40000$.
Evolving, $x=200$.
$\frac{3x}{5}=120$.

Contents of the larger, $200\times200=40,000$ square rods.
Contents of the smaller, $120\times120=14,400$ square rods.

Ex. (14.) Let x = the side of each square house-lot.
Then, $3x^2+193=25\times25=625$.
Uniting terms, &c., $3x^2=432$.
Dividing, $x^2=144$.
Evolving, $x=12$ rods each.

Ex. (15.) Let x^2 = the square rods in the field.
Then, $4x$ = rods round it.
And $\frac{x^2}{10}=4x$.
Clearing of fractions, $x^2=40x$.
Dividing by x, $x=40$, the side of the field.
Therefore, $40\times40=1600$ square rods.
$1600\div160=10$ acres.

Ex. (16.) Let x = the longer side, and $\frac{3x}{4}$ = the shorter.

Then, $x^2+\frac{9x^2}{16}=100\times100=10000$.

Clearing of fractions, $16x^2+9x^2=160000.$
Uniting terms, $25x^2=160000.$
Dividing, $x^2=6400.$
Evolving, $x=80.$

$$\frac{3x}{4}=60.$$

$80\times60=4800$; $4800\div160=30$ acres.

Ex. (17.) Let x = the number of days for which they were engaged.

Then, $x-4$ = the number A worked.
$x-7$ = the number B worked.
And $\frac{75}{x-4}$ = the number of shillings A received per day.
And $\frac{48}{x-7}$ = the number of shillings B received per day.

Then, $$\frac{75(x-7)}{x-4}=\frac{48(x-4)}{x-7}.$$

Reducing, &c., $25(x-7)^2=16(x-4)^2.$
Extracting square root, $5(x-7)=4(x-4).$
Reducing, &c., $5x-35=4x-16.$
Transposing, &c., $x=19$ days, the term for which they engaged to work.

$19-4=15$ days A worked.
$19-7=12$ days B worked.
$\frac{75}{15}=5$ shillings A received per day.
$\frac{48}{12}=4$ shillings B received per day.

Ex. (18.) Let x and $\frac{4x}{5}$ = the two numbers.

Then, $$x^3+\frac{64x^3}{125}=1512.$$

Clearing of fractions, $125x^3+64x^3=189000.$
Collecting terms, $189x^3=189000.$
Dividing, $x^3=1000.$
Extracting cube root, $x=10$, larger number.
And $\frac{4x}{5}=8$, less number.

Ex. (19.) Let x = the breadth of the box; then $3x$ = the length, and $\frac{3x}{4}$ = the height.

Therefore, $3x\times x\times\frac{3x}{4}=2150\frac{4}{5}\times50.$

Multiplying, $\frac{9x^3}{4}=107520.$

Clearing of fractions, $9x^3=430080.$

Dividing, $x^3=47786.666+.$

Extracting cube root, $x=36.28+$ inches breadth.

Multiplying, $3x=108.84+$ inches length.

And $\frac{3x}{4}=27.21+$ inches height.

Ex. (20.) Let x = the height, length and breadth.

Then, $x^3=2150\frac{4}{5}\times100=215040.$

Therefore, $x=59.9+$ inches, height, length and breadth.

Ex. (21.) Let x = the larger number, and $\frac{3x}{7}$ = the less.

Then, $x^3-\frac{27x^3}{343}=2528.$

Clearing of fractions, $343x^3-27x^3=867104.$

Collecting, $316x^3=867104.$

Dividing, $x^3=2744.$

Extracting the cube root, $x=14$, the larger.

And $\frac{3x}{7}=6$, the less number.

Ex. (22.) Let $3x$ = the length, and x = the breadth.

$3x\times x=3x^2$ square rods.

$3x^2\times x=5184.$

Multiplying, $3x^3=5184.$

Dividing, $x^3=1728.$

Extracting the cube root, $x=12$, breadth

$3x=36$, length.

Ex. (27.) $(68{,}000{,}000)^2=4{,}624{,}000{,}000{,}000{,}000.$

$(95{,}000{,}000)^2=9{,}025{,}000{,}000{,}000{,}000.$

That is, the intensity of light at Venus is to the intensity of light at the earth as 9025 to 4624.

Or, $4624 : 9025 :: 1 : x.$

$4624x=9025.$

$x=1.95+$ times.

That is, as 1.95+to 1.

$(7700)^2=59{,}290{,}000.$

$(7912)^2=62{,}599{,}744.$

$59{,}290{,}000\times9025=535{,}092{,}250{,}000.$

$62{,}599{,}744\times4624=289{,}461{,}216{,}256.$

That is, the quantity of light Venus receives from the sun is to the quantity the earth receives from it as 534,892,250,000 to 289,461,216,256.

Or, $289{,}461{,}216{,}256 : 535{,}092{,}250{,}000 :: 1 : x.$

$289{,}461{,}216{,}256x=535{,}092{,}250{,}000.$

$x=1.84+$ times.

Ex. (28.) $37{,}000{,}000^2 : 95{,}000{,}000^2 :: 1 : x.$

$1369{,}000{,}000{,}000{,}000 : 9025{,}000{,}000{,}000{,}000 :: 1 : x.$

$1369x=9025.$

$x=6\frac{811}{1369}$ times.

AFFECTED QUADRATIC EQUATIONS.

ART. 224. (p. 175.)

Ex. (3.) Given $3x^2+5x-8=34$ to find the values of x.

Conditions, $3x^2+5x-8=34.$

Transposing, $3x^2+5x=34+8=42.$

Dividing, $x^2+\frac{5x}{3}=14.$

Completing the square, $x^2+\frac{5x}{3}+\frac{25}{36}=14+\frac{25}{36}=\frac{529}{36}.$

Evolving, $x+\frac{5}{6}=\pm\frac{23}{6}.$

Transposing, $x=\pm\frac{23}{6}-\frac{5}{6}=3$, or $-4\frac{2}{3}.$

Ex. (4.) Given $x^2+6x+4=22-x$ to find the values of x.

Conditions, $x^2+6x+4=22-x.$
Transposing, $x^2+7x=18.$
Completing the square, $x^2+7x+\frac{49}{4}=18+\frac{49}{4}=\frac{121}{4}.$
Evolving, $x+\frac{7}{2}=\pm\frac{11}{2}.$
Transposing, $x=\pm\frac{11}{2}-\frac{7}{2}=2$, or -9.

Ex. (5.) Given $8x^2-7x+6=171$ to find the values of x.

Conditions, $8x^2-7x+6=171.$
Transposing, $8x^2-7x=165.$
Dividing, $x^2-\frac{7x}{8}=\frac{165}{8}.$
Completing the square, $x^2-\frac{7x}{8}+\frac{49}{256}=\frac{165}{8}+\frac{49}{256}=\frac{5329}{256}.$
Evolving, $x-\frac{7}{16}=\pm\frac{73}{16}.$
Transposing, $x=\pm\frac{73}{16}+\frac{7}{16}=5$, or $-\frac{33}{8}$.

Ex. (6.) Given $\frac{175x-350}{x}+10x-20=175$ to find the values of x.

Conditions, $\frac{175x-350}{x}+10x-20=175.$
Clearing of fractions, $175x-350+10x^2-20x=175x.$
Transposing, &c., $10x^2-20x=350.$
Dividing, $x^2-2x=35.$
Completing the square, $x^2-2x+1=35+1=36.$
Evolving, $x-1=\pm 6.$
Transposing, $x=\pm 6+1=7$, or -5.

Ex. (7.) Given $x^2-6x+12=4$ to find the values of x.

Conditions, $x^2-6x+12=4.$
Transposing, $x^2-6x=-8.$

Completing the square, $x^2-6x+9=-8+9=1.$
Evolving, $x-3=\pm 1.$
Transposing, $x=\pm 1+3=4, \text{ or } 2.$

ART. 225. (p. 183.)

Ex. (31.) Given $x=\sqrt{a^2+x\sqrt{b^2+x^2}}-a$ to find the value of x.

Conditions, $x=\sqrt{a^2+x\sqrt{b^2+x^2}}-a.$
Transposing, $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}.$
Involving, $x^2+2ax+a^2=a^2+x\sqrt{b^2+x^2}.$
Reducing, $x^2+2ax=x\sqrt{b^2+x^2}.$
Dividing, $x+2a=\sqrt{b^2+x^2}.$
Involving, $x^2+4ax+4a^2=b^2+x^2.$
Reducing, &c., $4ax=b^2-4a^2.$
Dividing, $x=\dfrac{b^2-4a^2}{4a}.$

Ex. (32.) Given $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$ to find the value of x.

Conditions, $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}.$
Clearing of fractions, $x^2-ax^2=x.$
Dividing, $x-ax=1.$
Reducing, &c., $x=\dfrac{1}{1-a}.$

Ex. (33.) Given $x^2+12x-16=92$ to find the values of x.
Conditions, $x^2+12x-16=92.$
Transposing, $x^2+12x=92+16=108.$
Completing the square, $x^2+12x+36=108+36=144.$
Evolving, $x+6=\pm 12.$
Transposing, $x=\pm 12-6=6, \text{ or } -18.$

Ex. (34.) Given $x^2-3x=10$ to find the values of x.

Conditions, $x^2-3x=10.$

Completing the square, $x^2-3x+2.25=10+2.25=12.25.$

Evolving, $x-1.5=\pm 3.5.$

Transposing, $x=\pm 3.5+1.5=5$, or -2.

Ex. (35.) Given $x^2-x+3=45$ to find the values of x.

Conditions, $x^2-x+3=45.$

Transposing, $x^2-x=45-3=42.$

Completing the square, $x^2-x+.25=42+.25=42.25.$

Evolving, $x-.5=\pm 6.5.$

Transposing, $x=\pm 6.5+.5=7$, or -6.

Ex. (36.) Given $5x^2+x=4$ to find the values of x.

Conditions, $5x^2+x=4.$

Dividing, $x^2+\frac{x}{5}=\frac{4}{5}.$

Completing the square, $x^2+\frac{x}{5}+\frac{1}{100}=\frac{4}{5}+\frac{1}{100}=\frac{81}{100}.$

Evolving, $x+\frac{1}{10}=\pm\frac{9}{10}.$

Transposing, $x=\pm\frac{9}{10}-\frac{1}{10}=\frac{4}{5}$, or -1.

Ex. (37.) Given $2x^2-x=21$ to find the values of x.

Conditions, $2x^2-x=21.$

Dividing, $x^2-\frac{x}{2}=\frac{21}{2}.$

Completing the square, $x^2-\frac{x}{2}+\frac{1}{16}=\frac{21}{2}+\frac{1}{16}=\frac{169}{16}.$

Evolving, $x-\frac{1}{4}=\pm\frac{13}{4}.$

Transposing, $x=\pm\frac{13}{4}+\frac{1}{4}=\frac{7}{2}$, or -3.

Ex. (38.) Given $5x^2+6x-3=60$ to find the values of x.

Conditions, $5x^2+6x-3=60.$

Transposing, $5x^2+6x=60+3=63.$

Dividing, $x^2+\frac{6x}{5}=\frac{63}{5}.$

Completing the square, $x^2+\frac{6x}{5}+\frac{36}{100}=\frac{63}{5}+\frac{36}{100}=\frac{1296}{100}.$

Evolving, $x+\frac{6}{10}=\pm\frac{36}{10}.$

Transposing, $x=\pm\frac{36}{10}-\frac{6}{10}=3$, or $-\frac{21}{5}$

Ex. (39.) Given $(x-12)(x+2)=0$ to find the values of x.

Conditions, $(x-12)(x+2)=0.$

Multiplying, $x^2-10x-24=0.$

Transposing, $x^2-10x=24.$

Completing the square, $x^2-10x+25=24+25=49.$

Evolving, $x-5=\pm 7.$

Transposing, $x=\pm 7+5=12$, or -2.

Ex. (40.) Given $3x^2-14x+15=0$ to find the values of x.

Conditions, $3x^2-14x+15=0.$

Transposing, $3x^2-14x=-15.$

Dividing, $x^2-\frac{14x}{3}=-5.$

Completing the square, $x^2-\frac{14x}{3}+\frac{196}{36}=-5+\frac{196}{36}=\frac{16}{36}.$

Evolving, $x-\frac{14}{6}=\pm\frac{4}{6}.$

Transposing, $x=\pm\frac{4}{6}+\frac{14}{6}=3$, or $1\frac{2}{3}$.

Ex. (41.) Given $ax^2-bx=c$ to find the values of x.

Conditions, $ax^2-bx=c.$

Dividing, $x^2-\frac{bx}{a}=\frac{c}{a}.$

Completing the square, $x^2-\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2+4ac}{4a^2}.$

Evolving, $x-\frac{b}{2a}=\frac{\pm\sqrt{b^2+4ac}}{2a}.$

Transposing, $x=\frac{b\pm\sqrt{b^2+4ac}}{2a}.$

Ex. (42.) Given $4x^2-6x=108$ to find the values of x.

Conditions, $4x^2-6x=108.$

Dividing, $x^2-\frac{3x}{2}=27.$

Completing the square, $x^2-\frac{3x}{2}+\frac{9}{16}=27+\frac{9}{16}=\frac{441}{16}.$

Evolving, $x-\frac{3}{4}=\pm\frac{21}{4}.$

Transposing, $x=\pm\frac{21}{4}+\frac{3}{4}=6$, or $-4\frac{1}{2}$.

Ex. (43.) Given $4x-\frac{14-x}{x+1}=14$ to find the values of x.

Conditions, $4x-\frac{14-x}{x+1}=14.$

Clearing of fractions, &c., $4x^2-9x=28.$

Dividing, $x^2-\frac{9x}{4}=7.$

Completing the square, $x^2-\frac{9x}{4}+\frac{81}{64}=7+\frac{81}{64}=\frac{529}{64}.$

Evolving, $x-\frac{9}{8}=\pm\frac{23}{8}.$

Transposing, $x=\pm\frac{23}{8}+\frac{9}{8}=4$, or $-\frac{7}{4}$.

Ex. (44.) Given $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$ to find the values of x.

Conditions, $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}.$

Clearing of fractions, $10x-14+2x=\frac{22x^2}{9}.$

" " " $90x-126+18x=22x^2.$

Transposing, $22x^2-108x=-126.$

Dividing, $x^2-\frac{54x}{11x}=-\frac{63}{11}.$

Completing the square, $x^2-\frac{54x}{11}+\frac{729}{121}=-\frac{63}{11}+\frac{729}{121}=\frac{36}{121}.$

Evolving, $x-\frac{27}{11}=\pm\frac{6}{11}.$

Transposing, $x=\pm\frac{6}{11}+\frac{27}{11}=3$, or $\frac{21}{11}.$

Ex. (45.) Given $x+\sqrt{5x+10}=8$ to find the values of x.

Conditions, $x+\sqrt{5x+10}=8.$
Transposing, $x-8=-\sqrt{5x+10}.$
Involving, $x^2-16x+64=5x+10.$
Transposing, &c., $x^2-21x=-54.$
Completing the square,
$x^2-21x+110.25=-54+110.25=56.25.$
Evolving, $x-10.5=\pm7.5.$
Transposing, $x=\pm7.5+10.5=18$, or $3.$

Ex. (46.) Given $x+\sqrt{10x+6}=9$ to find the values of x.

Conditions, $x+\sqrt{10x+6}=9.$
Transposing, $x-9=-\sqrt{10x+6}.$
Involving, $x^2-18x+81=10x+6.$
Reducing, $x^2-28x=-75.$
Completing the square, $x^2-28x+196=-75+196=121.$
Evolving, $x-14=\pm11.$
Transposing, $x=\pm11+14=25$, or $3.$

Ex. (47.) Given $3x^2+2x-9=76$ to find the values of x.

Conditions, $3x^2+2x-9=76.$
Transposing, $3x^2+2x=76+9=85.$

Dividing, $x^2+\frac{2x}{3}=\frac{85}{3}.$

Completing the square, $x^2+\frac{2x}{3}+\frac{1}{9}=\frac{85}{3}+\frac{1}{9}=\frac{256}{9}.$

Evolving, $x+\frac{1}{3}=\pm\frac{16}{3}.$

Transposing, $x=\pm\frac{16}{3}-\frac{1}{3}=5, \text{ or } -\frac{17}{3}.$

Ex. (48.) Given $x^2-10x=-25$ to find the value of x.

Conditions, $x^2-10x=-25.$
Transposing, &c., $x^2-10x+25=-25+25=0.$
Evolving, $x-5. \qquad 0.$
Transposing, $x=5.$

Ex. (49.) Given $3x^2-x-140=0$ to find the values of x.

Conditions, $3x^2-x-140=0.$
Transposing, $3x^2-x=140.$
Dividing, $x^2-\frac{x}{3}=\frac{140}{3}.$
Completing the square, $x^2-\frac{x}{3}+\frac{1}{36}=\frac{140}{3}+\frac{1}{36}=\frac{1681}{36}.$
Evolving, $x-\frac{1}{6}=\pm\frac{41}{6}.$
Transposing, $x=\pm\frac{41}{6}+\frac{1}{6}=7, \text{ or } -\frac{20}{3}.$

Ex. (50.) Given $5x^2+\frac{7x}{2}=7x^2-51$ to find the values of x

Conditions, $5x^2+\frac{7x}{2}=7x^2-51.$
Clearing of fractions, $10x^2+7x=14x^2-102.$
Transposing, &c., $4x^2-7x=102.$
Dividing, $x^2-\frac{7x}{4}=\frac{51}{2}.$
Completing the square, $x^2-\frac{7x}{4}+\frac{49}{64}=\frac{51}{2}+\frac{49}{64}=\frac{1681}{64}.$
Evolving, $x-\frac{7}{8}=\pm\frac{41}{8}.$
Transposing, $x=\pm\frac{41}{8}+\frac{7}{8}=6, \text{ or } -4\frac{1}{4}.$

Ex. (51.) Given $2x^2-\frac{4x-4}{3}=7x$ to find the values of x.

Conditions, $2x^2-\frac{4x-4}{3}=7x.$

Clearing of fractions, $6x^2-4x+4=21x.$

Transposing, &c., $6x^2-25x=-4.$

Dividing, $x^2-\frac{25x}{6}=-\frac{2}{3}.$

Completing the square, $x^2-\frac{25x}{6}+\frac{625}{144}=-\frac{2}{3}+\frac{625}{144}=\frac{529}{144}.$

Evolving, $x-\frac{25}{12}=\pm\frac{23}{12}.$

Transposing, $x=\pm\frac{23}{12}+\frac{25}{12}=4$, or $\frac{1}{6}$.

Ex. (52.) Given $\frac{x^2}{5}+20x=3x^2-80$ to find the values of x.

Conditions, $\frac{x^2}{5}+20x=3x^2-80.$

Clearing of fractions, $x^2+100x=15x^2-400.$

Transposing, &c., $14x^2-100x=400.$

Dividing, $x^2-\frac{50x}{7}=\frac{200}{7}.$

Completing the square, $x^2-\frac{50x}{7}+\frac{625}{49}=\frac{200}{7}+\frac{625}{49}=\frac{2025}{49}.$

Evolution, $x-\frac{25}{7}=\pm\frac{45}{7}.$

Transposing, $x=\pm\frac{45}{7}+\frac{25}{7}=10$, or $-2\frac{6}{7}$.

Ex. (53.) Given $x^2+8x=65$ to find the values of x.

Conditions, $x^2+8x=65.$

Completing the square, $x^2+8x+16=65+16=81.$

Evolving, $x+4=\pm 9.$

Transposing, $x=\pm 9-4=5$, or -13.

Ex. (54.) Given $6x^2-x=92$ to find the values of x.

Conditions, $6x^2-x=92$.

Dividing, $x^2-\frac{x}{6}=\frac{46}{3}$.

Completing the square, $x^2-\frac{x}{6}+\frac{1}{144}=\frac{46}{3}+\frac{1}{144}=\frac{2209}{144}$.

Evolving, $x-\frac{1}{12}=\pm\frac{47}{12}$.

Transposing, $x=\pm\frac{47}{12}+\frac{1}{12}=4$, or $-\frac{23}{6}$.

Ex. (55.) Given $3x^2+4x=340$ to find the values of x.

Conditions, $3x^2+4x=340$.

Dividing, $x^2+\frac{4x}{3}=\frac{340}{3}$.

Completing the square, $x^2+\frac{4x}{3}+\frac{16}{36}=\frac{340}{3}+\frac{16}{36}=\frac{4096}{36}$.

Evolving, $x+\frac{4}{6}=\pm\frac{64}{6}$.

Transposing, $x=\pm\frac{64}{6}-\frac{4}{6}=10$, or $-11\frac{1}{3}$.

Ex. (56.) Given $x^2-10x=-21$ to find the values of of x.

Conditions, $x^2-10x=-21$.

Completing the square, $x^2-10x+25=-21+25=4$.

Evolving, $x-5=\pm2$.

Transposing, $x=\pm2+5=7$, or 3.

Ex. (57.) Given $5x^2-\frac{x}{2}=78$ to find the values of x.

Conditions, $5x^2-\frac{x}{2}=78$.

Clearing of fractions, $10x^2-x=156$.

Dividing, $x^2-\frac{x}{10}=\frac{156}{10}$.

Completing the square, $x^2-\frac{x}{10}+\frac{1}{400}=\frac{156}{10}+\frac{1}{400}=\frac{6241}{400}$.

Evolving, $x-\frac{1}{20}=\pm\frac{79}{20}$.

Transposing, $x=\pm\frac{79}{20}+\frac{1}{20}=4$, or $-\frac{39}{10}$.

Ex. (58.) Given $11x^2-100x=-201$ to find the values of x.

Conditions, $11x^2-100x=-201$.

Dividing, $x^2-\frac{100x}{11}=-\frac{201}{11}$.

Completing the square, $x^2-\frac{100x}{11}+\frac{2500}{121}=-\frac{201}{11}+\frac{2500}{121}=\frac{289}{121}$.

Evolving, $x-\frac{50}{11}=\pm\frac{17}{11}$.

Transposing, $x=\pm\frac{17}{11}+\frac{50}{11}=6\frac{1}{11}$, or 3.

Ex. (59.) Given $3x^2-17x=2x^2+84$ to find the values of x.

Conditions, $3x^2-17x=2x^2+84$.
Transposing, $x^2-17x=84$.
Completing the square, $x^2-17x+72.25=84+72.25=156.25$.
Evolving, $x-8.5=\pm12.5$.
Transposing, $x=\pm12.5+8.5=21$, or -4.

Ex. (60.) Given $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$ to find the values of x.

Conditions, $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$.
Uniting terms, &c., $x+6=3\sqrt{x+16}$.
Involving, $x^2+12x+36=9x+144$.
Transposing, $x^2+3x=108$.
Completing the square, $x^2+3x+2.25=108+2.25=110.25$.
Evolving, $x+1.5=\pm10.5$.
Transposing, $x=\pm10.5-1.5=9$, or -12.

Ex. (61.) Given $9x+\sqrt{16x^2+36x^3}=15x^2-4$ to find the values of x.

Conditions, $9x+\sqrt{16x^2+36x^3}=15x^2-4$.

Transposing, $9x+4+\sqrt{16x^2+36x^3}=15x^2$.

Separating into factors, $9x+4+2x\sqrt{9x+4}=15x^2$.

Completing the square,

$$(9x+4)+2x\sqrt{9x+4}+x^2=15x^2+x^2=16x^2.$$

Evolving, $\sqrt{9x+4}+x=\pm 4x$.

Transposing, $\sqrt{9x+4}=\pm 4x-x=3x$, or $-5x$.

Involving, $9x+4=9x^2$, or $25x^2$.

If the former, we have $9x+4=9x^2$.

Transposing, $9x^2-9x=4$.

Dividing, $x^2-x=\frac{4}{9}$.

Completing the square, $x^2-x+\frac{1}{4}=\frac{4}{9}+\frac{1}{4}=\frac{25}{36}$.

Evolving, $x-\frac{1}{2}=\pm\frac{5}{6}$.

Transposing, $x=\pm\frac{5}{6}+\frac{1}{2}=\frac{4}{3}$, or $-\frac{1}{3}$.

But, if the latter, $9x+4=25x^2$.

Transposing, $25x^2-9x=4$.

Dividing, $x^2-\frac{9x}{25}=\frac{4}{25}$.

Completing the square, $x^2-\frac{9x}{25}+\frac{81}{2500}=\frac{4}{25}+\frac{81}{2500}=\frac{481}{2500}$.

Evolving, $x-\frac{9}{50}=\pm\frac{\sqrt{481}}{50}$.

Transposing, $x=\frac{9\pm\sqrt{481}}{50}$.

Ex. (62.) Given $x=\frac{12+8x^{\frac{1}{2}}}{x-5}$ to find the values of x.

Conditions, $x=\frac{12+8x^{\frac{1}{2}}}{x-5}$.

Clearing of fractions, $x^2-5x=12+8x^{\frac{1}{2}}$.

Transposing one x, $x^2-4x=12+8x^{\frac{1}{2}}+x$.

Completing the square, $x^2-4x+4=16+8x^{\frac{1}{2}}+x$.

Evolving, $x-2=\pm(4+x^{\frac{1}{2}})$.

Transposing, $x-x^{\frac{1}{2}}=6$.

Completing the square, $x-x^{\frac{1}{2}}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$.

Evolving, $x^{\frac{1}{2}}-\frac{1}{2}=\pm\frac{5}{2}$.

Transposing, $x^{\frac{1}{2}}=\pm\frac{5}{2}+\frac{1}{2}=3$, or -2.

Involving, $x=9$, or 4.

But, if $x-2=-4-x^{\frac{1}{2}}$.

Then, $x+x^{\frac{1}{2}}=-2$.

Completing the square, $x+x^{\frac{1}{2}}+\frac{1}{4}=-2+\frac{1}{4}=\frac{-7}{4}$.

Evolving, $x^{\frac{1}{2}}+\frac{1}{2}=\frac{\pm\sqrt{-7}}{2}$.

Transposing, $x^{\frac{1}{2}}=\frac{-1\pm\sqrt{-7}}{2}$.

Involving, $x=\frac{-3\pm\sqrt{-7}}{2}$.

Ex. (63.) Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$ to find the values of x.

Conditions, $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$.

Transposing, $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}-\frac{x^2}{a}=-\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}$.

Squaring both sides, $x^2-\frac{a^4}{x^2}+\frac{x^4}{a^2}-\frac{2x^2}{a}\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=a^2-\frac{a^4}{x^2}$.

Transposing, $x^2-\frac{2x^2}{a}\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\frac{x^4}{a^2}-a^2=0$.

Or, $x^2-\frac{2x}{a}(x^4-a^4)^{\frac{1}{2}}+\frac{x^4-a^4}{a^2}=0$.

Evolving, $$x-\frac{\sqrt{x^4-a^4}}{a}=0.$$

Transposing, $$x=\frac{\sqrt{x^4-a^4}}{a}.$$

Multiplying, $$ax=\sqrt{x^4-a^4}.$$

Involving, $$a^2x^2=x^4-a^4.$$

Transposing, $$x^4-a^2x^2=a^4.$$

Completing the square, $$x^4-a^2x^2+\frac{a^4}{4}=a^4+\frac{a^4}{4}=\frac{5a^4}{4}.$$

Evolving, $$x^2-\frac{a^2}{2}=\frac{\pm\sqrt{5}.a^2}{2}.$$

Transposing, $$x^2=\frac{1\pm\sqrt{5}.}{2}a^2.$$

Involving, $$x=\pm a\sqrt{\frac{1\pm\sqrt{5}}{2}}.$$

Ex. (64.) Given $x-1=2+\frac{2}{x^{\frac{1}{2}}}$ to find the values of x.

Conditions, $$x-1=2+\frac{2}{x^{\frac{1}{2}}}.$$

Since $$x-1=(x^{\frac{1}{2}}-1)(x^{\frac{1}{2}}+1).$$

Therefore, $$(x^{\frac{1}{2}}-1)(x^{\frac{1}{2}}+1)=2+\frac{2}{x^{\frac{1}{2}}}.$$

Dividing by $x^{\frac{1}{2}}+1$, $$x^{\frac{1}{2}}-1=\frac{2}{x^{\frac{1}{2}}}.$$

Clearing of fractions, $$x-x^{\frac{1}{2}}=2.$$

Completing the square, $$x-x^{\frac{1}{2}}+\frac{1}{4}=2+\frac{1}{4}=\frac{9}{4}.$$

Evolving, $$x^{\frac{1}{2}}-\frac{1}{2}=\pm\frac{3}{2}.$$

Transposing, $$x^{\frac{1}{2}}=\pm\frac{3}{2}+\frac{1}{2}=2,\text{ or }-1.$$

Involving, $$x=4,\text{ or }1.$$

Ex. (65.) Given $\sqrt[3]{x^3-a^3}=x-b$ to find the values of x.

Conditions, $\sqrt[3]{x^3-a^3}=x-b.$

Involving both sides, $x^3-a^3=x^3-3bx^2+3b^2x-b^3.$

Transposing and cancelling, $3bx^2-3b^2x=a^3-b^3.$

Dividing by $3b$, $x^2-bx=\frac{a^3-b^3}{3b}.$

Completing the square, $x^2-bx+\frac{b^2}{4}=\frac{a^3-b^3}{3b}+\frac{b^2}{4}=\frac{4a^3-b^3}{12b}.$

Evolving, $x-\frac{b}{2}=\sqrt{\frac{4a^3-b^3}{12}}.$

Transposing, $x=\pm\sqrt{\frac{a^3-b^3}{12}}+\frac{b}{2}.$

Ex. (66.) Given $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$ to find the values of x.

Conditions, $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}.$

Clearing of fractions, $\sqrt{4x^2}+2\sqrt{x}=16-x.$

Transposing, &c., $3x+2\sqrt{x}=16.$

Dividing, $x+\frac{2}{3}\sqrt{x}=\frac{16}{3}.$

Completing the square, $x+\frac{2}{3}\sqrt{x}+\frac{1}{9}=\frac{16}{3}+\frac{1}{9}=\frac{49}{9}.$

Evolving, $\sqrt{x}+\frac{1}{3}=\pm\frac{7}{3}.$

Transposing, $\sqrt{x}=\pm\frac{7}{3}-\frac{1}{3}=2$, or $-\frac{8}{3}.$

Involving, $x=4$, or $\frac{64}{9}.$

Ex. (67.) Given $\sqrt{x^3}-2\sqrt{x}-x=0\sqrt{x}$ to find the values of x.

Conditions, $\sqrt{x^3}-2\sqrt{x}-x=0\sqrt{x}.$

Dividing by $\sqrt{x}$, $x-2-\sqrt{x}=0.$

Transposing, $x-\sqrt{x}=2.$

Completing the square, $x-\sqrt{x}+\frac{1}{4}=2+\frac{1}{4}=\frac{9}{4}.$

Evolving, $\sqrt{x}-\frac{1}{2}=\pm\frac{3}{2}.$

Transposing, $\sqrt{x}=\pm\frac{3}{2}+\frac{1}{2}=2$, or $-1.$

Involving, $x=4$, or $1.$

Ex. (68.) Given $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$ to find the values of x.

Conditions, $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}.$

Dividing by $\sqrt{x}$, $x^2+x=6.$

Completing the square, $x^2+x+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}.$

Evolving, $x+\frac{1}{2}=\pm\frac{5}{2}.$

Transposing, $x=\pm\frac{5}{2}-\frac{1}{2}=2$, or -3

Ex. (69.) Given $\frac{x}{2}=22\frac{1}{6}+\frac{\sqrt{x}}{3}$ to find the values of x.

Conditions, $\frac{x}{2}=22\frac{1}{6}+\frac{\sqrt{x}}{3}.$

Multiplying by 2, $x=44\frac{1}{3}+\frac{2\sqrt{x}}{3}.$

Transposing, $x-\frac{2\sqrt{x}}{3}=44\frac{1}{3}.$

Completing the square, $x-\frac{2\sqrt{x}}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=\frac{400}{9}.$

Evolving, $\sqrt{x}-\frac{1}{3}=\pm\frac{20}{3}.$

Transposing, $\sqrt{x}=\pm\frac{20}{3}+\frac{1}{3}=7$, or $-\frac{19}{3}.$

Involving, $x=49$, or $\frac{361}{9}.$

Ex. (70.) Given $\frac{\frac{3\sqrt{x}}{5}-2}{x-5}-\frac{1}{20}=0$ to find the values of x.

Conditions, $\frac{\frac{3\sqrt{x}}{5}-2}{x-5}-\frac{1}{20}=0.$

Reducing the fraction, $\frac{3\sqrt{x}-10}{5x-25}=\frac{1}{20}.$

Clearing of fractions, $12\sqrt{x}-40=x-5.$

Transposing, $x-12\sqrt{x}=-35.$

Completing the square, $x-12\sqrt{x}+36=-35+36=1.$

Evolution, $\sqrt{x}-6=\pm1.$

Transposing, $\sqrt{x}=\pm1+6=7$, or 5.

Involving, $x=49$, or 25.

Ex. (71.) Given $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$ to find the values of x.

Conditions, $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756.$

Completing the square, $x^{\frac{6}{5}}+x^{\frac{3}{5}}+\frac{1}{4}=756+\frac{1}{4}=\frac{3025}{4}.$

Evolving, $x^{\frac{3}{5}}+\frac{1}{2}=\pm\frac{55}{2}.$

Transposing, $x^{\frac{3}{5}}=\pm\frac{55}{2}-\frac{1}{2}=27$, or -28.

Evolving, $x^{\frac{1}{5}}=3$, or $\sqrt[3]{-28}$.

Involving, $x=243$, or $-28^{\frac{5}{3}}$.

Ex. (72.) Given $x^3-x^{\frac{3}{2}}=56$ to find the values of x.

Conditions, $x^3-x^{\frac{3}{2}}=56.$

Completing the square, $x^3-x^{\frac{3}{2}}+\frac{1}{4}=56+\frac{1}{4}=\frac{225}{4}.$

Evolving, $x^{\frac{3}{2}}-\frac{1}{2}=\pm\frac{15}{2}.$

Transposing, $x^{\frac{3}{2}}=\pm\frac{15}{2}+\frac{1}{2}=8$, or -7.

Evolving, $x^{\frac{1}{2}}=2$, or $-7^{\frac{1}{3}}$.

Involving, $x=4$, or $-7^{\frac{2}{3}}$, or $\sqrt[3]{49}$.

Ex. (73.) Given $\sqrt{5+x}+\sqrt{x}=\frac{15}{\sqrt{5+x}}$ to find the values of x.

Conditions, $\sqrt{5+x}+\sqrt{x}=\frac{15}{\sqrt{5+x}}$.

Clearing of fractions, $5+x+\sqrt{5x+x^2}=15$.

Transposing, $\sqrt{5x+x^2}=10-x$.

Involving, $5x+x^2=100-20x+x^2$.

Reducing, $25x=100$.

Dividing, $x=4$.

Ex. (74.) Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$ to find the values of x.

Conditions, $\sqrt{x+12}+\sqrt[4]{x+12}=6$.

Completing the square, $\sqrt{x+12}+\sqrt[4]{x+12}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$.

Evolving, $\sqrt[4]{x+12}+\frac{1}{2}=\pm\frac{5}{2}$.

Transposing, $\sqrt[4]{x+12}=\pm\frac{5}{2}-\frac{1}{2}=2$, or -3

Involving, $x+12=16$, or 81

Transposing, $x=16-12=4$, or $81-12=69$.

Ex. (75.) Given $x^n-2ax^{\frac{n}{2}}=b$ to find the values of x.

Conditions, $x^n-2ax^{\frac{n}{2}}=b$.

Completing the square, $x^n-2ax^{\frac{n}{2}}+a^2=a^2+b$.

Evolving, $x^{\frac{n}{2}}-a=\pm\sqrt{a^2+b}$.

Transposing, $x^{\frac{n}{2}}=a\pm\sqrt{a^2+b}$.

Involving, &c., $x=(a\pm\sqrt{a^2+b})^{\frac{2}{n}}$.

Ex. (76.) Given $3x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592$ to find the values of x.

Conditions, $$3x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592.$$

Transposing, &c., $$\frac{5x^{\frac{8}{3}}}{2}-3x^{\frac{4}{3}}=592.$$

Multiplying and dividing, $$x^{\frac{8}{3}}-\frac{6x^{\frac{4}{3}}}{5}=\frac{1184}{5}.$$

Completing the square,

$$x^{\frac{8}{3}}-\frac{6x^{\frac{4}{3}}}{5}+\frac{9}{25}=\frac{1184}{5}+\frac{9}{25}=\frac{5929}{25}.$$

Evolving, $$x^{\frac{4}{3}}-\frac{3}{5}=\pm\frac{77}{5}.$$

Transposing, $$x^{\frac{4}{3}}=\pm\frac{77}{5}+\frac{3}{5}=16, \text{ or } -\frac{74}{5}.$$

Involving, &c., $$x=8, \text{ or } -\left(\frac{74}{5}\right)^{\frac{3}{4}}.$$

PROBLEMS IN QUADRATICS. (p. 187.)

Ex. (2.) Let $x=$ the width of the frame.

And $12\times18=216$, the contents of the glass.

Then $(18+12)(2\times x)=60x$; $x\times x\times4=4x^2$.

Hence $4x^2+60x=$ the contents of the frame.

And $4x^2+60x=216$.

Dividing, $$x^2+15x=54.$$

Completing the square,

$$x^2+15x+56.25=54+56.25=110.25.$$

Evolving, $$x+7.5=10.5.$$

Transposing, $$x=10.5-7.5=3 \text{ inches}.$$

Hence the width of the frame is 3 inches.

Ex. (3.) Let $x=$ the number of sheep.

Then, $$\frac{60\times20}{x}=\frac{54\times20}{x-15}-2.$$

Clearing of fractions, $1200x-18000=1080x-2x^2+30x.$

Transposing, &c., $2x^2+90x=\$18000.$

Dividing, $x^2+45x=9000.$

Completing the square,

$$x^2+45x+\frac{2025}{4}=9000+\frac{2025}{4}=\frac{38025}{4}.$$

Evolving, $$x+\frac{45}{2}=\frac{195}{2}.$$

Transposing, $$x=\frac{195}{2}-\frac{45}{2}=\frac{150}{2}=75 \text{ sheep}.$$

Dividing, $1200\div75=16$ shillings.

Hence there were 75 sheep, at 16 shillings each.

Ex. (4.) Let x = the price of the flour.

Then, $$\left(x\times\frac{x}{100}\right)+x=39.$$

And $$\frac{x^2}{100}+x=39.$$

Clearing of fractions, $x^2+100x=3900.$

Completing the square,

$$x^2+100x+2500=3900+2500=6400$$

Evolving, $x+50=80.$

Transposing, $x=80-50=\$30.$

Hence the flour cost \$30.

Ex. (5.) Let x = the less number, and $x+9$ = the larger.

Then $x+9+x=2x+9$ = the sum of their ages.

And $\overline{2x+9}\times\overline{x+9}=266.$

" $2x^2+27x+81=266.$

Uniting terms, &c., $$x^2+\frac{27x}{2}=92.5.$$

Completing the square, $$x^2+\frac{27x}{2}+\frac{729}{16}=92.5+\frac{729}{16}=\frac{2209}{16}.$$

Evolving, $x+\frac{27}{4}=\frac{47}{4}$.

Transposing, $x=\frac{47}{4}-\frac{27}{4}=5$, the less number.

" And $5+9=14$, the larger.

Hence the larger number is 14, and the less 5.

Ex. (6.) Let $x =$ the sum A put in the firm.

Then, as each man's gain will be in proportion to his stock multiplied by the time it was in trade, therefore,

$x\times 12=12x$ will represent A's gain.

And $30\times 16=480$ " " B's gain.

Then $12x+480 : 18 :: 12x : 26-x$.

Multiplying the extremes and means by each other, we have

$$312x+12480-480x-12x^2=216x.$$

Transposing, &c., $x^2+32x=1040$.

Completing the square, $x^2+32x+256=1040+256=1296$.

Evolving, $x+16=36$.

Transposing, $x=36-16=\$20$.

Hence A's capital was \$20.

[See National Arithmetic, page 226.]

Ex. (7.) Let $x =$ the number of barrels.

Then, $\frac{72}{x}=$ price per barrel.

Therefore, $\frac{72}{x+6}=\frac{72}{x}-1$.

Clearing of fractions, &c., $x+6x=432$.

Completing the square, $x^2+6x+9=432+9=441$.

Evolving, $x+3=21$.

Transposing, $x=21-3=18$ barrels.

Dividing, $72\div 18=\$4$, price of each.

Hence he bought 18 barrels, at \$4 per barrel.

Ex. (8.) Let $x =$ the width of the walk.

Then, $6x-2 =$ the side of the court.

And $8x-2 =$ the side of the court including the walk.

And $\overline{6x-2}\times4=24x-8$, perimeter of the walk.

Therefore, $(8x-2)^2-(6x-2)^2=24x-8+164$.

$(64x^2-32x+4)-(36x^2-24x+4=24x-8+164$.

Reducing, &c., $28x^2-32x=156$.

Dividing, $x^2-\frac{8x}{7}=\frac{39}{7}$.

Completing the square, $x^2-\frac{8x}{7}+\frac{16}{49}=\frac{39}{7}+\frac{16}{49}=\frac{289}{49}$.

Evolving, $x-\frac{4}{7}=\frac{17}{7}$.

Transposing, $x=\frac{17}{7}+\frac{4}{7}=\frac{21}{7}=3$, width of the walk.

$3\times6=18$; $18-2=16$, side of the court.

$16\times16=256$, area of the court.

Hence we find the court contains 256 square yards.

Ex. (9.) Given $\frac{1+x^3}{(1+x)^3}=\frac{1}{3}$ to find the values of x.

Conditions, $\frac{1+x^3}{(1+x)^3}=\frac{1}{3}$.

Involving, &c., $3+3x^3=1+3x+3x^2+x^3$.

Dividing by $1+x$, $3-3x+3x^2=1+2x+x^2$.

Reducing, $x^2-\frac{5x}{2}=-1$.

Completing the square, $x^2-\frac{5x}{2}+\frac{25}{16}=-1+\frac{25}{16}=\frac{9}{16}$.

Evolving, $x-\frac{5}{4}=\pm\frac{3}{4}$.

Transposing, $x=\pm\frac{3}{4}+\frac{5}{4}=2$, or $\frac{1}{2}$.

Ex. (10.) Given $x^4-2x^3+x=132$ to find the values of x.

Conditions, $x^4-2x^3+x=132$.

Adding and subtracting x^2, $x^4-2x^3+x^2-(x^2-x)=132$.

Completing the square,

$$(x^2-x)^2-(x^2-x)+\frac{1}{4}=132+\frac{1}{4}=\frac{529}{4}.$$

Evolving, $x^2-x-\frac{1}{2}=\pm\frac{23}{2}$.

Transposing, $x^2-x=\pm\frac{23}{2}+\frac{1}{2}=12$, or -11.

If we suppose the former, then, by completing the squares, we have

$$x^2-x+\frac{1}{4}=12+\frac{1}{4}=\frac{49}{4}.$$

Evolving, $x-\frac{1}{2}=\pm\frac{7}{2}$.

Transposing, $x=\pm\frac{7}{2}+\frac{1}{2}=4$, or -3.

But, if $x^2-x=-11$.

Completing the square, $x^2-x+\frac{1}{4}=-11+\frac{1}{4}=-\frac{43}{4}$.

Evolving, $x-\frac{1}{2}=\frac{\pm\sqrt{-43}}{2}$.

Transposing, $x=\frac{1\pm\sqrt{-43}}{2}$.

Ex. (11.) Given $9x+\sqrt{16x^2+36x^3}=15x^2-4$ to find the values of x.

Conditions, $9x+\sqrt{16x^2+36x^3}=15x^2-4$.

Transposing, &c., $(9x+4)+2x\sqrt{9x+4}=15x^2$.

Completing the square,

$$(9x+4)+2x\sqrt{9x+4}+x^2=15x^2+x^2=16x^2.$$

Evolving, $\sqrt{9x+4}+x=\pm4x$.

Therefore, $\sqrt{9x+4}=\pm3x$, or $-5x$.

Involving, $9x+4=9x^2$, or $25x^2$.

Transposing, $9x^2-9x=4$.

Completing the square, $9x^2-9x+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4}$.

Evolving, $3x-\frac{3}{2}=\pm\frac{5}{2}$.

Transposing, $3x=\pm\frac{5}{2}+\frac{3}{2}=4$, or -1.

Dividing, $x=\frac{4}{3}$, or $-\frac{1}{3}$.

But, if $9x+4=25x^2$.

Transposing, $25x^2-9x=4$.

Dividing, $x^2-9x=\frac{4}{25}$.

Completing the square,

$$x^2-\frac{9x}{25}+\frac{81}{2500}=\frac{4}{25}+\frac{81}{2500}=\frac{481}{2500}.$$

Evolving, $x-\frac{9}{50}=\frac{\pm\sqrt{481}}{50}$.

Transposing, $x=\frac{9\pm\sqrt{481}}{50}$.

Ex. (12.) Let $x=$ the first term, and $y=$ the second.

1. Then $x:y:y:16$.
2. Therefore, $16x=y^2$.
3. Again, $x^2+y^2=225$.
4. By substitution, $x^2+16x=225$.
5. Completing the square, $x^2+16x+64=225+64=289$.
6. Evolving, $x+8=17$.
7. Transposing, $x=17-8=9$.
8. Putting the value of x into (2), $16\times9=144=y^2$.
9. Evolving, $12=y$.

Hence the two numbers are 9 and 12.

QUADRATICS WITH TWO OR MORE UNKNOWN TERMS. (p. 188.)

Ex. (7.) $\left.\begin{array}{l}\text{Given } x^2+y^2=20\\ \text{And } x^2-y^2=12\end{array}\right\}$ to find the values of x and y.

1. First condition, $x^2+y^2=20$.
2. Second condition, $x^2-y^2=12$.
3. Subtracting (2) from (1), $2y^2=8$.
4. Dividing (3) by (2), $y^2=4$.
5. Evolving (4), $y=2$.
6. Substituting for y in the (2) its value in (5),

$$x^2-4=12.$$

7. Transposing (6), $x^2=12+4=16.$
8. Evolving, $x=4.$

Hence $x=4$; and $y=2$.

Ex. (8.) $\left.\begin{array}{l}\text{Given } x+y=6\\ \text{And } x^2+y^2=26\end{array}\right\}$ to find the values of x and y.

1. First condition, $x+y=6.$
2. Second condition, $x^2+y^2=26.$
3. Value of x in (1), $x=6-y.$
4. Involving (3), $x^2=36-12y+y^2.$
5. Putting (4) into (2), $36-12y+y^2+y^2=26.$
6. Transposing, &c. (5), $y^2-6y=-5.$
7. Completing the square, $y^2-6y+9=-5+9=4.$
8. Evolving, $y-3=\pm 2.$
9. Transposing, $y=\pm 2+3=5$, or 1.
10. Putting the value of y into (1), $x+5=6.$
11. Transposing, $x=6-5=1.$
12. Or, $x+1=6.$
13. Transposing, $x=6-1=5.$

Hence $x=5$, or 1; and $y=1$, or 5.

Ex. (9.) $\left.\begin{array}{l}\text{Given } x^2+y^2=74\\ \text{And } x-y=2\end{array}\right\}$ to find the values of x and y.

1. First condition, $x^2+y^2=74.$
2. Second condition, $x-y=2.$
3. Transposing (2), $x=2+y.$
4. Involving (3), $x^2=4+4y+y^2.$
5. Putting the value of x^2 into (1),
$4+4y+y^2+y^2=74.$
6. Dividing, &c., $y^2+2y=35.$
7. Completing the square, $y^2+2y+1=35+1=36$
8. Evolving, $y+1=\pm 6.$
9. Transposing, $y=\pm 6-1=5$, or -7.
10. Putting the value of y into (2), $x-5=2.$
11. Transposing, $x=2+5=7.$
12. Or, $x=2-7=-5.$

Hence $x=7$, or -5; and $y=5$, or -7.

Ex. (10.) $\left.\begin{array}{ll}\text{Given} & x^2+y^2=149 \\ \text{And} & x+y=17\end{array}\right\}$ to find the values of x and y.

1. First condition, $x^2+y^2=149.$
2. Second condition, $x+y=17.$
3. Transposing (2), $x=17-y.$
4. Involving (3), $x^2=289-34y+y^2$
5. Putting the value of x^2 into (1),

$$289-34y+y^2+y^2=149.$$

6. Reducing terms, $y^2-17y=-70.$
7. Completing the square,

$$y^2-17y+\frac{289}{4}=-70+\frac{289}{4}=\frac{9}{4}.$$

8. Evolving, $y-\frac{17}{2}=\pm\frac{3}{2}.$
9. Transposing, $y=\pm\frac{3}{2}+\frac{17}{2}=10, \text{ or } 7.$
10. Putting the value of y into (2),

$$x+10=17.$$

11. Transposing, $x=7.$
12. Or, putting the value y into (3), $x=17-7=10.$

Hence the value of $x=10$, or 7; $y=7$, or 10.

Ex. (11.) $\left.\begin{array}{ll}\text{Given} & x^2-y^2=85 \\ \text{And} & x+y=17\end{array}\right\}$ to find the values of x and y.

1. First condition, $x^2-y^2=85.$
2. Second condition, $x+y=17.$
3. Transposing (2), $x=17-y.$
4. Involving (3), $x^2=289-34y+y^2.$
5. Putting the value of x^2 into (1),

$$289-34y+y^2-y^2=85.$$

6. Reducing terms, $17y=102.$
7. Dividing, $y=6.$
8 Putting the value of y into (2), $x+6=17.$
9. Transposing, &c., $x=17-6=11.$

Hence the value of x is 11, and y is 6.

Ex. (12.) Given $\left.\begin{array}{l} x-y=2 \\ \text{And } x^3-y^3=98 \end{array}\right\}$ to find the values of x and y.

1. First condition, $x-y=2.$
2. Second condition, $x^3-y^3=98.$
3. Dividing (2) by (1), $x^2+xy+y^2=49.$
4. Involving (1), $x^2-2xy+y^2=4.$
5. Subtracting (4) from (3), $3xy=45.$
6. Dividing, $xy=15.$
7. Transposing (1), $x=2+y.$
8. Putting the value of x into (6), $y(2+y)=15.$
9. Multiplying, $y^2+2y=15.$
10. Completing the square, $y^2+2y+1=15+1=16.$
11. Evolving, $y+1=\pm 4.$
12. Transposing, $y=\pm 4-1=3$, or $-5.$
13. Putting the value of y into (7), $x=2+3=5.$
14. Or, $x=2-5=-3.$

Hence $x=5$, or -3; and $y=3$, or -5.

Ex. (13.) Given $\left.\begin{array}{l} 10x+y=3xy \\ \text{And } \quad y-x=2 \end{array}\right\}$ to find the values of x and y.

1. First condition, $10x+y=3xy.$
2. Second condition, $y-x=2.$
3. Transposing (2), $y=2+x.$
4. Putting the value of y into (1), $10x+2+x=3x(2+x).$
5. Reducing terms, &c., $x^2-\frac{5x}{3}=\frac{2}{3}.$
6. Completing the square,

$$x^2-\frac{5x}{3}+\frac{25}{36}=\frac{2}{3}+\frac{25}{36}=\frac{49}{36}.$$

7. Evolving, $x-\frac{5}{6}=\pm\frac{7}{6}.$
8. Transposing, $x=\pm\frac{7}{6}+\frac{5}{6}=2$, or $-\frac{1}{3}.$
9. Substituting 2 for its value in the (3), $y=2+2=4.$
10. Or, $y=2+\left(-\frac{1}{3}\right)=\frac{5}{3}.$

Hence $x=2$, or $-\frac{1}{3}$; and $y=4$, or $\frac{5}{3}$.

EXAMPLES OF ONE OR MORE UNKNOWN TERMS. (p. 192.)

Ex. (1.) Let x = A's money, and y = B's money.

1. Then, $x+y=18$.
2. And $2x\times y=2xy=154$.
3. Transposing, $x=18-y$
4. Putting x into (2), $2y(18-y)=154$.
5. Multiplying, $36y-2y^2=154$.
6. Transposing and dividing, $y^2-18y=-77$.
7. Completing the square,
$$y^2-18y+81=-77+81=4.$$
8. Evolving, $y-9=2$.
9. Transposing, $y=2+9=11$.
10. Putting the value of y into (1), $x+11=18$.
11. Reducing, $x=7$.

Hence A had $7, and B $11.

Ex. (2.) Let x = the larger number, and y = the less.

1. Then, $x-y=5$.
2. And $x^2+y^2=193$.
3. Involving (1), $x^2-2xy+y^2=25$.
4. Subtracting (3) from (2), $2xy=168$.
5. Dividing, $xy=84$.
6. Transposing (1), $x=5+y$.
7. Putting the value of x into (5), $y(5+y)=84$.
8. Multiplying, $5y+y^2=84$.
9. Completing the square, $y^2+5y+\frac{25}{4}=84+\frac{25}{4}=\frac{361}{4}$.
10. Evolving, $y+\frac{5}{2}=\frac{19}{2}$.
11. Transposing, $y=\frac{19}{2}-\frac{5}{2}=\frac{14}{2}=7$.
12. Putting the value of y into (1), $x-7=5$.
13. Transposing, $x=5+7=12$.

Hence 12 = the larger, and 7 = the smaller number.

Ex. (3.) As it requires 200 rods of fence to enclose both fields, it is evident that the length of one side of each will be $\frac{200}{4}=50$ rods. Let x = side of A's, and y = side of B's field.

1. Therefore, $x+y=50.$
2. And $x^2+y^2=1300.$
3. Transposing (1), $x=50-y.$
4. Involving (1), $(x+y)^2=x^2+2xy+y^2=2500.$
5. Subtracting (2) from (4), $2xy=1200.$
6. Dividing, $xy=600.$
7. Substituting for x its value in the (3), $y(50-y)=600.$
8. Multiplying, $50y-y^2=600.$
9. Transposing, $y^2-50y=-600.$
10. Completing the square,
    $$y^2-50y+625=-600+625=25.$$
11. Evolving, $y-25=5.$
12. Transposing, $y=5+25=30$ rods.
13. Substituting, $x=50-30=20$ rods.
14. $30\times30=900$; $900\times2.25=\$20.25$, B's.
15. $20\times20=400$; $400\times2.25=\$9$, A's.

Ex. (4.) Let x = the side of the smaller room, and $x+1$ the side of the larger.

1. Then, $x^2+\overline{x+1}^2=85.$
2. And $x^2+x^2+2x+1=85.$
3. Reducing, $x^2+x=42.$
4. Completing the square, $x^2+x+\frac{1}{4}=42+\frac{1}{4}=\frac{169}{4}.$
5. Evolving, $x+\frac{1}{2}=\frac{13}{2}.$
6. Transposing, $x=\frac{13}{2}-\frac{1}{2}=6$, side of the smaller room.
7. By question, $6+1=7$ = side of the larger.
8. " $40 : 6^2 :: 6^2 : 32\frac{2}{5}$ yards.
9. " $1.75\times32\frac{2}{5}=\$56.70$, price for the smaller room.
10. " $40 : 6^2 :: 7^2 : 44\frac{1}{10}$ yards.
11. " $1.75\times44\frac{1}{10}=\$77.17\frac{1}{2}$, price for the larger room.

Ex. (5.) Let x = the side of the larger pile, and $20-x$ = the side of the smaller.

1. Then, by conditions, $x^3+\overline{20-x}^3=2240$.
2. Involving, $x^3+8000-1200x+60x^2-x^3=2240$.
3. Reducing terms, $60x^2-1200x+8000=2240$.
4. Transposing, $x^2-20x=-96$
5. Completing square, $x^2-20x+100=-96+100=4$.
6. Evolving, $x-10=2$.
7. Transposing, $x=2+10=12$, side of the larger.
8. By conditions, $20-12=8$, side of the less.
9. “ $12^3=1728$.
10. “ $8^3=512$.
11. “ $1728\div128=13\frac{1}{2}$ cords.
12. “ $512\div128=4$ cords.
13. “ $\$6.25\times13\frac{1}{2}=\$84.37\frac{1}{2}$, value of the larger pile
14. “ $\$6.25\times4=\$25$, value of the less.

Ex. (6.) Let x = the length of the larger building, and y = the length of the less; and $\frac{48}{4}=12$, the difference of their lengths.

1. Then, by second condition, $x^2+y^2=2120$.
2. By first condition, $x-y=12$.
3. Involving (2), $x^2-2xy+y^2=144$.
4. Subtracting (3) from (1), $2xy=1976$.
5. Dividing, $xy=988$.
6. Transposing (2), $x=12+y$.
7. Putting the value of the x into the (5), $y(12+y)=988$.
8. Multiplying, $12y+y^2=988$.
9. Completing the square, &c.,
   $y^2+12y+36=988+36=1024$.
10. Evolving, $y+6=32$.
11. Transposing, $y=32-6=26$, length of the less.
12. “ $x=12+26=38$, length of the larger.

Ex. (7.) Let x = the number of hours each was on the road before they met.

1. Then, $x+9$=A's whole time.
2. And $x+16$=B's whole time.

3. Then, $x+9 : x :: 1 : \frac{x}{x+9}$ = distance A goes before they meet.
4. And $x+16 : x :: 1 : \frac{x}{x+16}$ = distance B goes.
5. Hence $\frac{x}{x+9}+\frac{x}{x+16}=1$, whole distance.
6. Clearing fractions, $x^2+16x+x^2+9x=x^2+25x+144$.
7. Therefore, $x^2=144$.
8. Evolving, $x=12$.
9. By the question, $x+9=12+9=21$ hrs., A's time.
10. " " " $x+16=12+16=28$ hrs., B's time.

Hence A performs his journey in 21 hours, and B in 28 hours.

Ex. (8.) Let x = the larger part.
And $60-x$ = the smaller.

1. Then, $x\times\overline{60-x} : x^2-\overline{60-x}^2 :: 2 : 3$.
2. Mult. extremes, &c., $180x-3x^2=-7200+240x$.
3. Reducing terms, &c., $x^2+20x=2400$.
4. Completing square, $x^2+20+100=2400+100=2500$.
5. Evolving, $x+10=50$.
6. Transposing, $x=50-10=40$, the larger.
7. " $60-40=20$, the less.

Hence 40 = the larger part, and 20 = the smaller.

Ex. (9.) Let x = the larger, and y = the less number.

1. Then, by first condition, $xy=77$.
2. And $y=\frac{77}{x}$.
3. By 2d condition, $x^2-\left(\frac{77}{x}\right)^2 : \left(x-\frac{77}{x}\right)^2 :: 9 : 2$.
4. Involving, &c., $2\left(x^2-\frac{5929}{x^2}\right)=\left(x^2-154+\frac{5929}{x^2}\right)9$.
5. Multiplying, $2x^4-11858=9x^4-1386x^2+53361$.
6. Reducing terms, $x^4-198x^2=-9317$.
7. Completing the square,
$$x^4-198x^2+9801=-9317+9801=484.$$

8. Evolving, $x^2-99=22$.
9. Transposing, $x^2=22+99=121$.
10. Evolving, $x=11$, larger number.
11. Substituting, $y=\frac{77}{11}=7$, less number.

Hence the larger number $=11$, and the smaller $=7$.

Ex. (10.) 1. First condition, $x^2+y^2=225$
2. Second condition, $x^2 : y^2 :: 9 : 16$.
3. Multiplying extremes, &c., $16x^2=9y^2$.
4. Dividing, $x^2=\frac{9y^2}{16}$.
5. Substituting for x^2 in (1) its value in (4), $y^2+\frac{9y^2}{16}=225$.
6. Adding terms, $\frac{25y^2}{16}=225$.
7. Evolving, $\frac{5y}{4}=15$.
8. Dividing, $y=12$.
9. Involving, $y^2=144$.
10. Value of x^2 in (1), $x^2=225-144=81$.

Hence the less lot contained 81 square rods, and the larger 144 square rods.

Ex. (11.) Let x = the greater, and y = the less.
1. By the question, $xy=48$.
2. And $x^3-y^3 : (x-y)^3 :: 37 : 1$.
3. Expanding, $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: 37 : 1$.
4. By Prop. IX.,* $3x^2y-3xy^2 : (x-y)^3 :: 36 : 1$.
5. Dividing into factors, $3xy(x-y) : (x-y)^3 :: 36 : 1$.
6. Divide by $x-y$, $3xy : (x-y)^2 :: 36 : 1$.
7. $xy=48$ (1), therefore, $144 : (x-y)^2 :: 36 : 1$.
8. Multiplying extremes, &c., $36(x-y)^2=144$.
9. Dividing by 36, $(x-y)^2=4$.
10. Evolution, $x-y=2$.
11. Involving (10), $x^2-2xy+y^2=4$.
12. Multiplying (1) by 4, $4xy=192$.

---

* p. 210 Algebra.

13. Adding (12) to (11), $x^2+2xy+y^2=196.$
14. Evolving (13), $x+y=14.$
15. Evolving (11), $x-y=2.$
16. Sum of (14) and (15), $2x=16.$
17. Dividing, $x=8.$
18. Value of y in (14), $y=14-8=6.$

Hence the greater number is 8, and the less 6.

NOTE. — This question might have been solved if $x+y$ = the greater, and $x-y$ = the less.

Ex. (12.) Let x = the greater, and y = the less.

1. Then, by first condition, $xy=196.$
2. Second condition, $\frac{x}{y}=4.$
3. Multiplying, $x=4y.$
4. Substituting for x its value in (3), $4y\times y=196.$
5. Multiplying, $4y^2=196.$
6. Dividing, $y^2=49.$
7. Evolving, $y=7$, the less.
8. Multiplying, $4\times7=28$, the greater.

Hence the greater number is 28, and the less 7.

Ex. (13.) Let x = the time in which they all can do the work; then, in one hour, A will do $\frac{x}{6}$ of it, B $\frac{x}{15}$, and C $\frac{x}{10}$ of it.

1. Therefore, $\frac{x}{6}+\frac{x}{15}+\frac{x}{10}=1$ hour.
2. Adding, $\frac{x}{3}=1.$
3. Multiplying, $x=3$ hours.

A, B and C, will therefore do the work in 3 hours.

Ex. (14.) Let x = the number of oxen.

1. Then, $\frac{240}{x}$ = the price of one ox.
2. And $\overline{x-3}\times\overline{\frac{240}{x}+8}=240+59.$

3. Clearing of fractions, &c.,

$$8x^2+216x-720=299x.$$

4. Transposing, $8x^2-83x=720.$

5. Dividing, $x^2-\frac{83x}{8}=90.$

6. Completing the square,

$$x^2-\frac{83x}{8}+\frac{6889}{256}=90+\frac{6889}{256}=\frac{29929}{256}.$$

7. Evolving, $x-\frac{83}{16}=\frac{173}{16}.$

8. Transposing, $x=\frac{173}{16}+\frac{83}{16}=\frac{256}{16}=16$ oxen.

Hence the grazier bought 16 oxen.

Ex. (15.) Let x = the side of the less, and y = the side of the greater.

1. Then, by first condition, $x+y=41.$

2. Second condition, $x^2\times\frac{y}{4}+y^2\times\frac{x}{4}=205\times 20=4100.$

3. Multiplying, &c., $x^2y+y^2x=16400.$

4. Dividing (3) by (1), $xy=400.$

5. Dividing by y, $x=\frac{400}{y}.$

6. Substituting for x in (1) its value in (5),

$$\frac{400}{y}+y=41.$$

7. Multiplying by y, $400+y^2=41y.$

8. Transposing, $y^2-41y=-400.$

9. Completing the square,

$$y^2-41y+\frac{1681}{4}=-400+\frac{1681}{4}=\frac{81}{4}$$

10. Evolving, $y-\frac{41}{2}=\frac{9}{2}.$

11. Transposing, $y=\frac{9}{2}+\frac{41}{2}=25$, side of the larger.

12. Substituting, $41-25=16$, side of the less.

Ex. (16.) Let x = the larger part, and $145-x$ = the smaller.

1. Then, by conditions, $\sqrt{x}+\sqrt{145-x}=17$.
2. Involving, $x+2\sqrt{145x-x^2}+145-x=289$.
3. Collecting terms, $2\sqrt{145x-x^2}=144$.
4. Dividing, $\sqrt{145x-x^2}=72$.
5. Involving, $145x-x^2=5184$.
6. Transposing, $x^2-145x=-5184$.
7. Completing square, $x^2-145x+\frac{20925}{4}=-5184+\frac{20925}{4}$.
8. " " $x^2-145x+\frac{20925}{4}=\frac{289}{4}$.
9. Evolving, $x-\frac{145}{2}=\frac{17}{2}$.
10. Transposing, $x=\frac{17}{2}+\frac{145}{2}=\frac{162}{2}=81$, larger.
11. " $145-81=64$, smaller.

Hence the larger number is 81, and the smaller 64.

Ex. (17.) Let x = the sum paid for the ox.

1. Then, by conditions, $x+\left(x\times\frac{x}{100}\right)=56$.
2. Clearing of fractions, &c., $x^2+100x=5600$.
3. Completing square, $x^2+100x+2500=5600+2500=8100$.
4. Evolving, $x+50=90$.
5. Transposing, $x=90-50=40$.

Hence the sum paid was $40.

Ex. (18.) Let x = the larger part, and $14-x$ = the smaller

1. Then, by conditions, $x^3+(14-x)^3=728$.
2. Expanding, $x^3+2744-588x+42x^2-x^3=728$.
3. Reducing terms, $42x^2-588x=-2016$.
4. Dividing, $x^2-14x=-48$.
5. Completing the square, $x^2-14x+49=-48+49=1$.
6. Evolving, $x-7=1$.
7. Transposing, $x=1+7=8$, the larger.
8. " $14-8=6$, the smaller.

Hence the numbers are 8 and 6.

Ex. (19.) Let x = the breadth, and $2x$ = the length of the field.

1. Then, by conditions, $x\times\overline{2x}=\overline{x+4}\times\overline{2x+4}-496.$
2. Transposing, &c., $2x^2+496=2x^2+12x+16.$
3. Reducing terms, $12x=480.$
4. Dividing, $x=40.$
5. Multiplying, $2x=80.$
6. " $\overline{40+4}\times\overline{80+4}=3696.$
7. Dividing, $3696\div160=23$ acres, 16 poles.

Ex. (20.) Let x and y = the numbers.

1. Then, by first condition,
$$xy+x^2+y^2=109.$$
2. By 2d condition, $x^2-y^2=24.$
3. Transposing, $x^2=24+y^2.$
4. Evolving, $x=\sqrt{24+y^2}.$
5. Substituting for x in the (1) its value in (3) and (4),
$$y\sqrt{24+y^2}+24+y^2+y^2=109.$$
6. Transposing, $y\sqrt{24+y^2}=85-2y^2.$
7. Involving (6), $24y^2+y^4=7225-340y^2+4y^4.$
8. Transposing, $3y^4-364y^2=-7225.$
9. Dividing, $y^4-\frac{364y^2}{3}=-\frac{7225}{3}.$
10. Completing the square,
$$y^4-\frac{364y^2}{3}+\frac{132496}{36}=-\frac{7225}{3}+\frac{132496}{36}=\frac{45796}{36}.$$
11. Evolving, $y^2-\frac{364}{6}=\pm\frac{214}{6}.$
12. Transposing, $y^2=\pm\frac{214}{6}+\frac{364}{6}=\frac{150}{6}=25,$
13. " $y=5.$
14. Involving, &c., $y^2=25 : 24+25=49.$
15. Putting the value of y^2 into (3),
$$x^2=24+25=49.$$
16. Evolving, $x=\sqrt{49}=7.$

Hence the numbers are 5 and 7.

Ex. (21.) Let x = the number.

Then, by the conditions, $\sqrt{x+40}=x-16$.

Involving, $x+40=x^2-32x+256$.

Transposing, $x^2-33x=-216$.

Completing the square, $x^2-33x+\frac{1089}{4}=-216+\frac{1089}{4}=\frac{225}{4}$.

Evolving, $x-\frac{33}{2}=\frac{15}{2}$.

Transposing, $x=\frac{15}{2}+\frac{33}{2}=\frac{48}{2}=24$.

Hence the number is 24.

Ex. (22.) Let x = A's age, and y = B's age.

1. By first condition, $xy=750$.
2. By 2d condition, $\overline{x-2}\times\overline{y+7}=851$.
3. Multiplying, $xy+7x-2y-14=851$.
4. Subtract. (1) from (3), $7x-2y=115$.
5. Dividing (1) by y, $x=\frac{750}{y}$.
6. Putting the value of x in (5) into (4),

$$\frac{5250}{y}-2y=115.$$

7. Clearing fractions, $5250-2y^2=115y$.
8. Transp. and dividing, $y^2+\frac{115y}{2}=2625$.
9. Completing the square,

$$y^2+\frac{115y}{2}+\frac{13225}{16}=2625+\frac{13225}{16}=\frac{55225}{16}.$$

10. Evolving, $y+\frac{115}{4}=\frac{235}{4}$.
11. Transp., $y=\frac{235}{4}-\frac{115}{4}=\frac{120}{4}=30$.
12. Dividing, $750\div30=25$.

Hence A's age is 25 years, and B's 30 years.

Ex. (23.) Let x = the length, and y = the breadth, of the garden.

1. First condition, $xy=15000.$
2. Second condition, $\overline{x+14}\times\overline{y+14}=18696.$
3. Multiplying, $xy+14x+14y+196=18696.$
4. Subtracting (1) from (3), $14x+14y+196=3696.$
5. Transposing, $14x+14y=3500.$
6. Value of x in (1), $x=\dfrac{15000}{y}.$
7. Putting value of x into (5), $\dfrac{210000}{y}+14y=3500.$
8. Clearing of fractions, $210000+14y^2=3500y.$
9. Dividing by 14, $15000+y^2=250y.$
10. Transposing, $y^2-250y=-15000.$
11. Completing the square,
$$y^2-250y+15625=-15000+15625=625.$$
12. Evolving, $y-125=25.$
13. Transposing, $y=25+125=150$ yards.
14. Dividing, $15000\div150=100$ yards.

Hence the length is 150, and the breadth 100 yards.

Ex. (24.) Let x = the number of acres.

Then, $\dfrac{5600}{x}$ = the price per acre.

Therefore, by cond., $\dfrac{5600}{x+10}=\dfrac{5600}{x}-10.$

Clearing of fractions, $5600x=5600x+56000-10x^2-100x.$

Transposing and cancelling,
$$10x^2+100x=56000.$$

Dividing, $x^2+10x=5600.$

Completing the square,
$$x^2+10x+25=5600+25=5625.$$

Evolving, $x+5=75.$

Transposing, $x=75-5=70$ acres.

Hence the farm consisted of 70 acres.

Ex. (25.) Let x = the breadth of the field, and $4x$ = the length.

Then, by cond., $x\times 4x=4x^2=$ the contents in rods.

And $\frac{4x^2}{160}=$ the contents in acres.

Therefore, $\frac{4x^2}{160}\times x=\frac{4x^3}{160}$, the cost of the field.

Distance round the field, $\overline{4x+x}\times 2=10x$.

And, by conditions, $10x\times 4=40x=\frac{4x^3}{160}$.

Therefore, $6400x=4x^3$.
Dividing by x, $6400=4x^2$.
Dividing by 4, $1600=x^2$.
Transposing and evolving, $x=40$.
Multiplying, $4x=160$.
" $160\times 40=6400$ square rods.
Dividing, $6400\div 160=40$ acres.
Multiplying, $40\times 40=\$1600$.

Hence, the length 160, and the breadth 40 rods. The price, $1600.

Ex. (26.) Let $x=$ the miles B travelled per hour, and $x+\frac{1}{4}=$ the miles A travelled per hour.

By conditions, $\frac{39}{x}-\frac{39}{x+\frac{1}{4}}=1$.

Clearing of fractions, $39x+\frac{39}{4}-39x=x^2+\frac{x}{4}$.

Reducing terms, $\frac{39}{4}=x^2+\frac{x}{4}$.

Completing the square, $x^2+\frac{x}{4}+\frac{1}{64}=\frac{39}{4}+\frac{1}{64}=\frac{625}{64}$.

Evolving, $x+\frac{1}{8}=\frac{25}{8}$.

Transposing, $x=\frac{25}{8}-\frac{1}{8}=\frac{24}{8}=3$ miles.

Substituting, $3+\frac{1}{4}=3\frac{1}{4}$ miles.

Hence A travelled $3\frac{1}{4}$ miles per hour, and B 3 miles.

Ex. (27.) Let x = the larger number, and y = the less.

1. By conditions, $(x-y)y=42$.
2. Multiplying, $xy-y^2=42$.
3. Transposing and dividing, $x=\frac{42+y^2}{y}$.
4. Involving (3), $x^2=\frac{1764+84y^2+y^4}{y^2}$.
5. By conditions, $(x+y)(x-y)=x^2-y^2=133$.
6. Substituting for x^2 its value in (4),
$$\frac{1764+84y^2+y^4}{y^2}-y^2=133.$$
7. Clearing of fractions,
$$1764+84y^2+y^4-y^4=133y^2.$$
8. Transposing, $49y^2=1764$.
9. Dividing by 49, $y^2=36$.
10. Evolving, $y=6$.
11. Value of x in (3), $x=\frac{42+36}{6}=13$.

Hence the larger number 13, and the less 6.

Ex. (28.) Let x = the number of persons.

And, by conditions, $\frac{6300}{x}$ = each man's share.

1. By conditions, $\frac{6300}{x-2}=\frac{6300}{x}+200$.
2. Clearing fract., $6300x=6300x-12600+200x^2-400x$.
3. Canc. &c., $200x^2-400x=12600$.
4. Dividing, $x^2-2x=63$.
5. Completing the square,
$$x^2-2x+1=63+1=64.$$
6. Evolving, $x-1=8$.
7. Transposing, $x=8+1=9$ persons.

Ex. (29.) Let x = the number of hills in breadth, $x+75$ = the number in length.

1. By conditions, $x\times\overline{x+75}=x^2+75x=6250$.
2. Completing the square,
$$x^2+75x+\frac{5625}{4}=6250+\frac{5625}{4}=\frac{30625}{4}.$$

3. Evolving, $x+\frac{75}{2}=\frac{175}{2}$.

4. Transposing, $x=\frac{175}{2}-\frac{75}{2}=\frac{100}{2}=50$ hills in breadth.

5. Substituting, $50+75=125$ hills in length.

Ex. (30.) Let x = the price of a duck, and y = the price of a turkey.

1. By first condition, $10x+12y=22.50$.
2. By second condition, $\frac{6}{x}-\frac{5}{y}=4$.
3. Clearing of fractions, $6y-5x=4xy$.
4. Mult. (3) by 2, $-10x+12y=8xy$.
5. Subtracting (4) from (1), $20x=22.50-8xy$.
6. Transposing (1), $x=\frac{22.50-12y}{10}$.
7. Putting x into (5), $45.00-24y=22.50-\frac{180.00y-96y^2}{10}$.
8. Mult. by 10, $450.00-240y=225.00-180.00y+96y^2$.
9. Reducing terms, $96y^2+60y=225$.
10. Dividing by 96, $y^2+\frac{5y}{8}=\frac{75}{32}$.
11. Completing the square,
$$y^2+\frac{5y}{8}+\frac{25}{256}=\frac{75}{32}+\frac{25}{256}=\frac{625}{256}.$$
12. Evolving, $y+\frac{5}{16}=\frac{25}{16}$.
13. Transposing, $y=\frac{25}{16}-\frac{5}{16}=\frac{20}{16}=1.25$.
14. Putting the value of y into (2),
$$\frac{6}{x}-\frac{5}{1.25}=4.$$
15. Clearing of fractions, $7.50-5x=5x$.
16. Transposing, $10x=7.50$.
17. Dividing, $x=.75$ cents.

Hence a duck costs 75 cents, and a turkey $1.25.

Ex. (31.) Let $x=$ the number.

1. By conditions, $\sqrt{x+24}=x-18$.
2. Involving, $x+24=x^2-36x+324$.
3. Reducing terms, $x^2-37x=-300$.
4. Completing the square,

$$x^2-37x+\frac{1369}{4}=-300+\frac{1369}{4}=169.$$

5. Evolving, $x-\frac{37}{2}=\frac{13}{2}$.
6. Transposing, $x=\frac{13}{2}+\frac{37}{2}=25$.

Ex. (32.) Let x and $y=$ one side of each garden.

1. By first condition, $x^2+y^2=208$.
2. By second condition, $4x+4y=80$.
3. Value of x in the (2), $x=\frac{80-4y}{4}=20-y$.
4. Putting the value of x into (1), $(20-y)^2+y^2=208$.
5. Expanding the (4), $400-40y+y^2+y^2=208$.
6. Transposing, &c., $y^2-20y=-96$.
7. Completing the square,

$$y^2-20y+100=-96+100=4.$$

8. Evolving, $y-10=2$.
9. Transposing, $y=2+10=12$.
10. Involving, $y^2=144$.
11. Subtracting, $208-144=64$.

Hence the larger garden contains 144 square rods, and the smaller 64.

Ex. (33.) Let $x=$ one of the sides of the larger garden, and $y=$ the smaller.

1. By first condition, $x^2-y^2=80$.
2. By second condition, $4x+4y=80$.
3. Value of x in (2), $x=\frac{80-4y}{4}=20-y$.
4. Involving (3), $x^2=400-40y+y^2$.
5. Putting value of x^2 into (1), $400-40y+y^2-y^2=80$.

6. Reducing terms, $40y=320.$
7. Dividing, $y=8.$
8. Involving (7), $y^2=64.$
9. Putting the value of y into (1), $x^2-64=80.$
10. Transposing, $x^2=144.$
11. Adding, $144+64=208.$

Hence the gardens contain 208 square rods.

Ex. (34.) Let $x+4$ = the side of the larger garden, and x = the side of the smaller.

1. By conditions, $(x+4)^2+x^2=208.$
2. Expanding, $x^2+8x+16+x^2=208.$
3. Reducing terms, $x^2+4x=96.$
4. Completing the square, $x^2+4x+4=96+4=100.$
5. Evolving, $x+2=10.$
6. Transposing, $x=10-2=8.$
7. By conditions, $8+4=12.$
8. Involving, $12^2+8^2=144+64=208.$
9. Transposing, $144-64=80.$

Hence the larger garden exceeds the smaller by 80 square rods.

Ex. (35.) Let x = a side of the smaller block, and y = a side of the larger.

1. By first condition, $x+y=20.$
2. By second condition, $x^3+y^3=2240.$
3. Value of x in (1), $x=20-y.$
4. Involving (3), $x^3=8000-1200y+60y^2-y^3.$
5. Value of (2), $8000-1200y+60y^2-y^3+y^3=2240.$
6. Reducing terms, $60y^2-1200y=-5760.$
7. Dividing, $y^2-20y=-96.$
8. Completing the square,
$$y^2-20y+100=-96+100=4.$$
9. Evolving, $y-10=2.$
10. Transposing, $y=2+10=12.$
11. Substituting, $x=20-12=8.$
12. " $12\times12\times6=864;\ 8\times8\times6=384.$

Hence the surface of the larger is 864 inches, and the smaller 384 inches.

Ex. (36.) Let x = the price of the cloth.

1. By conditions, $x+x\times\frac{x}{100}=75.$
2. Clearing of fractions, $100x+x^2=7500.$
3. Transposing, $x^2+100x=7500.$
4. Completing the square,
$x^2+100x+2500=7500+2500=10000$
5. Evolving, $x+50=100.$
6. Transposing, $x=100-50=50.$

Hence the price of the cloth was $50.

Ex. (37.) Let x = the larger number, and $x-12$ = the less.

1. By conditions, $\overline{2x-12}\times x=2x^2-12x=560.$
2. Dividing, &c., $x^2-6x=280.$
3. Completing the square, $x^2-6x+9=280+9=289.$
4. Evolving, $x-3=17.$
5. Transposing, $x=17+3=20.$
6. Substituting, $20-12=8.$

Hence the larger number is 20, and the less 8.

Ex. (38.) Let x = twice the width of the frame.

1. By conditions, $\overline{36+x}\times\overline{12+x}=\overline{36\times12}+448.$
2. Multiplying, &c., $432+48x+x^2=880.$
3. Reducing, $x^2+48x=448.$
4. Completing square, $x^2+48x+576=448+576=1024$
5. Evolving, $x+24=32.$
6. Transposing, $x=32-24=8$ inches.
7. Dividing, $8\div2=4$ inches

Hence the width of the frame is 4 inches.

Ex. (39.) Let the parts be x and $100-x$.

1. By conditions, $\sqrt{x}+\sqrt{100-x}=14.$
2. Involving $x+2\sqrt{100x-x^2}+100-x=196.$
3. Reducing, $2\sqrt{100x-x^2}=96.$
4. Dividing by 2, $\sqrt{100x-x^2}=48.$
5. Involving, $100x-x^2=2304.$

6. Transposing, $x^2-100x=-2304.$
7. Completing the square,
$$x^2-100x+2500=-2304+2500=196.$$
8. Evolving, $x-50=14.$
9. Transposing, $x=14+50=64.$
10. Substituting, $100-64=36.$

Hence the parts are 64 and 36.

Ex. (40.) Let x = the width of the gravel-walk.
And $6x-1$ = one side of the court.
Then $\overline{7x-1}\times 4x=28x^2-4x$ = contents of the walk.
And $\overline{6x-1}\times 4=24x-4$ = perimeter of the court.
Therefore, $28x^2-4x=24x+336.$

1. By conditions, $28x^2-28x=336.$
2. Dividing, $x^2-x=12.$
3. Completing the square, $x^2-x+\frac{1}{4}=12+\frac{1}{4}=12.25.$
4. Evolving, $x-\frac{1}{2}=3\frac{1}{2}.$
5. Transposing, $x=3\frac{1}{2}+\frac{1}{2}=4.$
6. Substituting, $6x-1=24-1=23.$
7. Involving, $23^2=529.$

Hence the area of the court is 529 square yards, and the width of the walk is 4 yards.

Ex. (41.) Let $2x$ = the number of bushels of barley.
Then x = the number of shillings paid for a bushel of wheat.
And $54x$ = the price paid for all the wheat.
And $x-4$ = the number of shillings paid for a bushel of barley.
Then $\overline{x-4}\times 2x=2x^2-8x$ = the price of all the barley.
Therefore, $2x^2-8x+54x=10(54+2x)+576.$

1. Transposing, $2x^2+26x=1116.$
2. Dividing, $x^2+13x=558.$
3. Comp. square, $x^2+13x+\frac{169}{4}=558+\frac{169}{4}=\frac{2401}{4}.$
4. Evolving, $x+\frac{13}{2}=\frac{49}{2}.$

5. Transposing, $x=\frac{49}{2}-\frac{13}{2}=18.$
6. Multiplying, $2x=36$ bushels of barley.
7. Substituting, $x-4=18-4=14$ shillings.

Hence there were 36 bushels of barley, at 14 shillings per bushel.

Ex. (42.) Let x = the side of the box, and $165\frac{1}{2}\times144\times3=71496$, solid contents of the plank.

1. Then, $x^3-(x-6)^3=71496.$
2. Expanding, $x^3-(x^3-18x^2+108x-216)=71496.$
3. Reducing terms, $x^2-6x=3960.$
4. Completing the square, $x^2-6x+9=3960+9=3969.$
5. Evolving, $x-3=63.$
6. Transposing, $x=63+3=66.$
7. Subtracting, $66-6=60$ inches$=5$ feet.
8. Involving, $5^3=125$ cubic feet.

Ex. (43.) Let x = the inside diameter, and $x+.1$ = the whole diameter.

1. By conditions, $(x+.1)^3-x^3=1.$
2. Expanding, $x^3+.3x^2+.03x+.001-x^3=1.$
3. Cancelling, &c., $.3x^2+.03x=1-.001=.999.$
4. Dividing by .3, $x^2+.1x=3.33.$
5. Completing the square,
$$x^2+.1x+.0025=3.33+.0025=3.3325.$$
6. Evolving, $x+.05=1.825+.$
7. Transposing, $x=1.825-.05=1.775+$ inches.
8. Adding, $1.775+.1=1.875+$ inches.

Hence the inside diameter is 1.775+ inches, and the whole diameter is 1.875+ inches.

Ex. (44.) Let x = the length of one box, and $20-x$ = the other.

1. Conditions, $(20-x)^3+x^3=2240.$
2. Expanding, $8000-1200x+60x^2-x^3+x^3=2240.$
3. Transposing and reducing, $x^2-20x=-96.$

4. Comp. the square, $x^2-20x+100=-96+100=4.$
5. Evolving, $x-10=2.$
6. Transposing, $x=2+10=12.$
7. Subtracting, $20-12=8.$
8. Involving, $12^3=1728;\ 8^3=512.$
9. Subtracting, $1728-512=1216.$

Hence the difference of the contents is 1216 cubic inches.

Ex. (45.) Let x = the side of one lot, and y = the side of the other.

1. First condition, $x^2+y^2=6100.$
2. Second condition, $x^2-y^2=1100.$
3. Subtracting (2) from (1), $2y^2=5000.$
4. Dividing, $y^2=2500.$
5. Evolving, $y=50.$
6. Subtracting, $6100-2500=3600.$
7. Evolving, $\sqrt{3600}=60.$

Hence the side of one lot is 60 feet, and the side of the other 50.

Ex. (46.) Let x = A's acres.
$200-x$ = B's acres.
$y+.75$ = A paid per acre.
y = B paid per acre.

1. By first condition, $x\times\overline{y+.75}=200.$
2. By second condition, $\overline{200-x}\times y=200.$
3. Reducing (1), $xy+.75x=200.$
4. Reducing (2), $200y-xy=200.$
5. Value of y in (3), $y=\dfrac{200-.75x}{x}.$
6. Substituting for y in the (4) for its value in (5),

$$200\frac{(200-.75x)}{x}-x\frac{(200-.75x)}{x}=200.$$

7. Clearing of fractions,

$$(40{,}000-150x)-(200x-.75x^2)=200x.$$

8. Reducing terms, $.75x^2-550x=-40{,}000.$

9. Dividing by .75, $x^2-\frac{2200x}{3}=-53333\frac{1}{3}$.

10. Completing the square,

$$x^2-\frac{2200x}{3}+\frac{1210000}{9}=-53333\frac{1}{3}+\frac{1210000}{9}=\frac{730000}{9}.$$

11. Evolving, $x-\frac{1100}{3}=\pm\frac{854.4}{3}$.

12. Transposing,

$$x=\pm\frac{854.4}{3}+\frac{1100}{3}=81.866+\text{, A's acres.}$$

13. Subtracting, $200-81.866+=118.133+$, B's acres.

14. Reducing terms,

$$y=\frac{200-75x}{x}=\frac{200-(.75\times81.866)}{81.866}=\$1.693+\text{, B paid per acre.}$$

15. Reducing terms,

$$y+.75=1.693+.75=\$2.443\text{, A paid per acre.}$$

Hence A had 81.866+ acres, at \$2.443+ per acre; B had 118.133+ acres, at \$1.693+ per acre.

Ex. (47.) £4 10*s.*=1080*d.*; 3*s.* 9*d.*=45*d.*

Let x = the time in which B can do the work.

Then, $9 : x :: 45 : \frac{45x}{9}=5x$, the sum which C must receive from A in part payment for his labor. Then, $45+5x$ = the money received by C for his 2 days' labor.

Now, it is evident that the sum received by C must bear the same proportion to the sum received for the whole work as the part of the work which he performs bears to the whole work. Therefore, $1080 : 45+5x :: 1$ work $: \frac{45+5x}{1080}=\frac{x+9}{216}$; and $\frac{x+9}{216}$ denotes the part of the work performed by C.

Therefore, $\frac{x+9}{216}$ work : 1 work :: 2 days : $\frac{432}{x+9}$ days = the time in which C could do the whole work.

Now, since by the question and operation we see that A performs $\frac{5}{9}$ of the work, B $\frac{5}{x}$ of it, and C $\frac{x+9}{216}$ of it, it is evident that $\frac{5}{9}+\frac{5}{x}+\frac{x+9}{216}=1$ work.

And, by transposition, $\frac{5}{x}+\frac{x+9}{216}=1-\frac{5}{9}=\frac{4}{9}$.

Clearing fractions, $1080+x^2+9x=96x$.

Transposing, $x^2-87x=-1080$.

Comp. the square, $x^2-87x+\frac{7569}{4}=-1080+\frac{7569}{4}=\frac{3249}{4}$.

Evolution, $x-\frac{87}{2}=\pm\frac{57}{2}$.

Transposing, $x=\pm\frac{57}{2}+\frac{87}{2}=15$ days, or 72 days.

" $\frac{432}{x+9}=\frac{432}{24}=18$ days.

NOTE.—The last value of x is excluded, from the nature of the question. We therefore find that B would reap the field in 15 days, and C in 18 days.

CUBIC AND HIGHER EQUATIONS.

ART. 228. (p. 201.)

Ex. (3.) Find the value of x in the equation $x^3+10x^2+5x=260$.

Let x = 4 and 5.

First Supposition.		Second Supposition.
64	x^3	125
160	$10x^2$	250
20	$5x$	25
244	Sums.	400
260		260
-16	Errors.	$+140$

Sum of the errors, $16+140=156$.
As $156 : 1 :: 16 : .1$.
$4+.1=4.1$ nearly.

Again, let $x = 4.1$ and 4.2.

First Supposition.		Second Supposition.
68.921	x^3	74.088
168.1	$10x^2$	176.4
20.5	$5x$	21.
257.521	Sums.	271.488
260.		260.
-2.479	Errors.	$+11.488$

Sum of the errors, $2.479+11.488=13.967$.
As $13.967 : .1 :: 2.479 : .017$.
Hence $x=4.1+.017=4.117+$ nearly.

Ex. (4.) Find the value of x in the equation $x^3-2x=50$.

Let $x = 3.8$ and 3.9.

First Supposition.		Second Supposition.
54.872	x^3	59.319
-7.6	$-2x$	-7.8
47.272	Sums.	51.519
50		50
-2.728	Errors.	1.519

Sum of the errors, $2.728+1.519=4.247$.
As $4.247 : .1 :: 2.728 : .064+$.
Hence $3.8+.064=3.864+$ nearly.

Ex. (5.) Find the value of x in the equation $x^4-3x^2-75x=10000$.

Let $x = 10$ and 11.

First Supposition.		Second Supposition.
10000	x^4	14641
-300	$-3x^2$	-363
-750	$-75x$	-825
8950	Sums.	13453
10000		10000
-1050	Errors.	$+3453$

Sum of the errors, $1050+3453=4503$.
As $4503 : 1 :: 1050 : .2+$.
Hence $x=10+.2=10.2+$ nearly.

Ex. (6.) Find the value of x in the equation $x^5+2x^4+3x^3+4x^2+5x=54321$.

Let $x=8.4$ and 8.5.

First Supposition.				Second Supposition.
41821.19424	=	x^5	=	44370.53125
9957.4272	=	$2x^4$	=	10440.1250
1778.112	=	$3x^3$	=	1842.375
282.24	=	$4x^2$	=	289.00
42.0	=	$5x$	=	42.5
53880.97344		Sums.		56984.53125
54321				54321
—440.02656		Errors.		+2663.53125

Sum of the errors, $440.02656+2663.53125=3103.557781$.
As $3103.557781 : .1 :: 440.02656 : .014$.
Hence $8.4+.014=8.414+$ nearly.

Ex. (7.) Conditions, $x^3+6x^2=432$.

By trial we perceive that x lies between 5 and 7. We therefore assume its value to be 5.9 and 6.1.

First Supposition.				Second Supposition.
205.379	=	x^3	=	226.981
208.86	=	$6x^2$	=	223.26
414.239	=	Sums.	=	450.241
432				432
—17.761	=	Errors.	=	+18.241

Sum of the errors, $17.761+18.241=36.002$.
Then, $36.002 : .2 :: 17.761 : .1$ nearly.
$5.9+.1=6$. *Ans.*
Proof, $6^3+6\times6^2=432$.

Ex. (8.) Conditions, $5x^3-10x^2=45$.

We find by trial that the value of x is between 2 and 4, and we assume its values to be 2.9 and 3.1.

First Supposition.				Second Supposition.
121.945	=	$5x^3$	=	148.955
−84.1	=	$-10x^2$	=	−96.1
−37.845	=	Sums.	=	52.855
45				45
−7.155	=	Errors.	=	+7.855

Sum of the errors, $7.155+7.855=15.01$.
$15.01 : .2 :: 7.155 : .1$ nearly.
$2.9+.1=3$. *Ans.*
Proof, $5(3)^3-10(3)^2=45$.

Ex. (9.) Conditions, $x^4-10x^2=375$.

We find by trial that the value of x is between 4.9 and 5.1.

First Supposition.				Second Supposition.
576.4801	=	x^4	=	676.5201
−240.1	=	$-10x^2$	=	−260.1
336.3801	=	Sums.	=	416.4201
375				375
−38.6199	=	Errors.	=	+41.4201

Sum of the errors, $38.6199+41.4201=80.04$.
$80.04 : .2 :: 38.6199 : .1$ nearly.
$4.9+.1=5$. *Ans.*
Proof, $(5)^4-10(5)^2=45$.

PROBLEMS FOR PROPORTION. (p. 217.)

Ex. (4.) Let x = the larger number; then $\frac{3x}{5}$ = the smaller

1. Therefore, $x+4 : \frac{3x}{5}+8 :: 6 : 5$.
2. Multiplying extremes, &c., $5x+20=\frac{18x}{5}+48$.
3. Clearing of fractions, $25x+100=18x+240$.
4. Reducing terms, $7x=140$.
5. Dividing, $x=20$, the larger.
6. Dividing, &c., $\frac{3\times20}{5}=12$, the smaller.

Hence the larger number is 20, and the smaller 12.

Ex. (5.) Let $x =$ the larger part, and $60-x =$ the smaller.

1. Then, $x\times\overline{60-x} : x^2-(60-x)^2 :: 2 : 3$.
2. Mult. extremes, &c., $180x-3x^2=-7200+240x$.
3. Reducing terms, &c., $x^2+20x=2400$.
4. Completing square, $x^2+20x+100=2400+100=2500$.
5. Evolving, $x+10=50$.
6. Transposing, $x=50-10=40$, the larger.
7. Subtracting, $60-40=20$, the smaller.

Hence the larger number is 40, and the smaller 20.

Ex. (6.) Let $x =$ the side of the larger lot, and $y =$ the side of the less.

1. Then, by first condition, $x^2+y^2=208$.
2. Transposing, $x^2=208-y^2$.
3. By second condition, $208-y^2 : y^2 :: 9 : 4$.
4. Multiplying extremes, &c., $832-4y^2=9y^2$.
5. Transposing, $13y^2=832$.
6. Dividing, $y^2=64$, smaller.
7. Putting the value of y^2 into (1), $x^2+64=208$.
8. Reducing, $x^2=208-64=144$, larger.
9. Subtracting, $144-64=80$ sq. rods.

Hence the larger has 80 square rods more than the smaller.

Ex. (7.) Let $x =$ the larger of the two numbers, and $y =$ the smaller.

1. Then, by first condition, $xy=12$.
2. And, by second condition, $x^3-y^3 : (x-y)^3 :: 13 : 4$.
3. Expanding, $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: 13 : 4$.
4. By Prop. IX., $3x^2y-3xy^2 : (x-y)^3 :: 9 : 4$.
5. Dividing by $x-y$, $3xy : (x-y)^2 :: 9 : 4$.
6. Value of three times xy in (1), $36 : (x-y)^2 :: 9 : 4$.
7. Multiplying extremes, &c., $144=9(x-y)^2$.
8. Dividing, $16=(x-y)^2$.
9. Evolving and transposing, $x-y=4$.
10. First condition, $xy=12$.
11. Transposing (9), $x=4+y$.
12. Putting the value of x in (11) into (10),
    $y(4+y)=12$.

13. Multiplying, $y^2+4y=12.$
14. Completing the square, $y^2+4y+4=12+4=16.$
15. Evolving, $y+2=4.$
16. Transposing, $y=4-2=2.$
17. Putting the value of y into (11), $x=4+2=6.$

Hence the numbers are 2 and 6.

Ex. (8.) Let x = the larger part, and $100-x$ = the smaller.

1. Then, by conditions,

$$6\times x\times\overline{100-x} : x^2+\overline{100-x}^2 :: 24 : 17.$$

2. Expanding, &c.,

$$600x-6x^2 : x^2+10000-200x+x^2 :: 24 : 17.$$

3. Multiplying extremes, &c.,

$$10200x-102x^2=24x^2+240000-4800x+24x^2.$$

4. Reducing terms, &c.,

$$x^2-100x=-1600.$$

5. Completing the square,

$$x^2-100x+2500=-1600+2500=900.$$

6. Evolving, $x-50=30.$
7. Transposing, $x=30+50=80.$
8. Subtracting, $100-80=20.$

Hence the two parts are 80 and 20.

Ex. (9.) Let x = the larger number, and y = the smaller.

1. Then, by first condition, $xy=35$, and $y=\frac{35}{x}.$
2. And, by 2nd condition, $x^2-\left(\frac{35}{x}\right)^2 : \left(x-\frac{35}{x}\right)^2 :: 6 : 1.$
3. Expanding second term,

$$x^2-\frac{1225}{x^2} : x^2-70+\frac{1225}{x^2} :: 6 : 1.$$

4. Mult. extremes, &c., $x^2-\frac{1225}{x^2}=6x^2-420+\frac{7350}{x^2}.$
5. Clearing of fractions, $x^4-1225=6x^4-420x^2+7350.$
6. Reducing terms, $x^4-84x^2=-1715.$
7. Completing the square,

$$x^4-84x+1764=-1715+1764=49.$$

8. Evolving, $x^2-42=7.$
9. Transposing, $x^2=7+42=49.$
10. Evolving, &c., $x=7;\ y=\frac{35}{7}=5.$

Hence the two numbers are 7 and 5.

Ex. (10.) Let x = the larger, and y = the smaller.

1. Then, by first condition, $x : y :: 4^3 : 1^3 :: 64 : 1.$
2. Multiplying extremes, $x=64y.$
3. By the second condition, $\sqrt{xy}=32.$
4. Therefore, $\sqrt{64y^2}=32.$
5. Involving, $64y^2=(32)^2=1024.$
6. Dividing, $y^2=16.$
7. Evolving, $y=4.$
8. By first condition, $x=64y.$
9. Therefore, $x=64\times4=256.$

Hence the numbers are 256 and 4.

Ex. (11.) Let x = the larger number, and $20-x$ = the smaller.

1. Then, $\frac{x}{20-x} : \frac{20-x}{x} :: 9 : 4.$
2. Clearing of fractions, $x^2 : 400-40x+x^2 :: 9 : 4.$
3. Reducing, &c., $5x^2-360x=-3600.$
4. Dividing, $x^2-72x=-720.$
5. Completing the square,
$$x^2-72x+1296=-720+1296=576.$$
6. Evolving, $x-36=\pm24.$
7. Transposing, $x=\pm24+36=12$, the larger.
8. Subtracting, $20-12=8$, the smaller.

NOTE.—By the conditions of the question we must consider ±24 only -24.

Ex. (12.) Let x = the first term, y = the second and third, and $9x$ = the fourth term.

1. Then, $x : y :: y : 9x.$
2. By Prop. I., $y^2=9x^2.$

3. Evolving, $y=3x.$
4. The terms, $x+3x+9x=26.$
5. And $13x=26.$
6. Dividing, $x=2.$
7. By substitution, $y=3\times2=6.$
8. By conditions, $9x=9\times2=18.$

Hence the three terms are 2, 6 and 18.

ARITHMETICAL PROGRESSION.

Art. 263. (p. 223.)

Ex. (13.) $L=a+(n-1)d=8+(10-1)5=53.$ Ans.

Ex. (14.) $d=\dfrac{L-a}{n-1}=\dfrac{1325-500}{12-1}=75.$

Ex. (15.) $L=a+(n-1)-d=12+(10-1)-3=-15$

Art. 268. (p. 226.)

Ex. (18.) $d=\dfrac{2S-2an}{n(n-1)}=\dfrac{(2\times285)-(2\times5\times15)}{15(15-1)}=2$

Ex. (19.) $d=\dfrac{2S-2an}{n(n-1)}=\dfrac{2(399)-(2\times3\times19)}{19(19-1)}=2.$

Ex. (20.) $d=\dfrac{2S-2an}{n(n-1)}=\dfrac{2(100)-(2\times7\times8)}{8(8-1)}=1\frac{4}{7}$

Problems. (p. 226.)

Ex. (1.) $L=a+(n-1)d=5+(7-1)3=23.$

Ex. (2.) $L=a+(n-1)d=3+(5-1)4\frac{1}{3}=20\frac{1}{3}.$

Ex. (3.) $L=a+(n-1)d=18+(7-1)\frac{1}{4}=19\frac{1}{2}.$

Ex. (4.) $L.=a+(n-1)d=7+(5-1)2\frac{1}{2}=17.$

Ex. (5.) $L=a+(n-1)d=\frac{3}{4}+(10-1)\frac{4}{5}=7\frac{19}{20}.$

Ex. (6.) $L.=a+(n-1)d=0+(20-)1\frac{1}{2}=28\frac{1}{2}$.

Ex. (7.) $L=a+(n-1)-d=10+(4-1).-2=4$.

Ex. (8.) $L.=-a+(n-1)-d=-8+(10-1)-3=-35$.

Ex. (9.) $L.=a+(n-1)-d=85+(10-1).-7=22$.

Ex. (10.) $L=a+(n-1)d=3\frac{1}{3}+(5-1)2\frac{1}{4}=12\frac{1}{3}$.

$$S=\left(\frac{L+a}{2}\right)n=\left(\frac{12\frac{1}{3}+3\frac{1}{3}}{2}\right)5=39\frac{1}{6}.$$

Ex. (11.) $L=a+(n-1)-d=2\frac{1}{2}+(10-1).-\frac{1}{4}=\frac{1}{4}$.

$$S=\left(\frac{L+a}{2}\right)n=\left(\frac{\frac{1}{4}+2\frac{1}{2}}{2}\right)10=13\frac{3}{4}.$$

Ex. (12.) $n=\left(\frac{L-a}{d}+1\right)=(dn=L-a+d)=(L-a=dn-d)=(L-a=(n-1)d)L=a+(n-1)d$.

Ex. (13.) $S=\left(\frac{L+a}{2}\right)n=\left(\frac{99+1}{2}\right)50=2500$, sum of terms.

Ex. (14.) $L=a+(n-1)d=4\frac{1}{2}+(8-1)3\frac{1}{2}=29$, last term.

$S=\left(\frac{L+a}{2}\right)n=\left(\frac{29+4\frac{1}{2}}{2}\right)8=134$, sum of series.

Ex. (15.) $L=a+(n-1)-d=7+(6-1)-4=-13$, last t'm.

$S=\left(\frac{L+a}{2}\right)n=\left(\frac{-13+7}{2}\right)6=-18$, sum of series.

Ex. (16.) $d=\frac{L-a}{n-1}=\frac{19-5}{6-1}=2\frac{4}{5}$, common difference.

$5+2\frac{4}{5}=7\frac{4}{5}$; $7\frac{4}{5}+2\frac{4}{5}=10\frac{3}{5}$; $10\frac{3}{5}+2\frac{4}{5}=13\frac{2}{5}$; $13\frac{2}{5}+2\frac{4}{5}=16\frac{1}{5}$.

Hence the terms of progression are $7\frac{4}{5}$, $10\frac{3}{5}$, $13\frac{2}{5}$, and $16\frac{1}{5}$.

Ex. (17.) $d=\frac{L-a}{n-1}=\frac{18-(-9)}{5-1}=6\frac{3}{4}$.

$-9+6\frac{3}{4}=-2\frac{1}{4}$; $-2\frac{1}{4}+6\frac{3}{4}=4\frac{1}{2}$; $4\frac{1}{2}+6\frac{3}{4}=11\frac{1}{4}$.

Hence the terms of progression are $-2\frac{1}{4}$, $4\frac{1}{2}$, and $11\frac{1}{4}$.

Ex. (18.) $a=L-(n-1)d=20-(8-1)5=-15$, first term.

$$S=\left(\frac{L+a}{2}\right)n=\left(\frac{20+(-15)}{2}\right)8=20, \text{ sum series.}$$

Ex. (19.) Let the three digits be $\overline{x-y}$, x, $\overline{x+y}$.

Then $\overline{x-y}+x+\overline{x+y}=12$.
Collecting terms, $3x=12$.
Dividing, $x=4$.
Therefore,

$$100(4-y)+\overline{10\times4}+\overline{4+y}+396=100(4+y)+\overline{10\times4}+\overline{4-y}.$$

And,

$$400-100y+40+4+y+396=400+100y+40+4-y.$$

Collecting terms, $840-99y=444+99y$.
Transposing, $198y=396$.
Dividing, $y=2$.

$$4-2=2;\ 4;\ 4+2=6.$$

Hence the three digits are 2, 4, and 6; and the number therefore is 246.

Ex. (20.) $8\times10=80$ miles, the distance A travels.

$L=2+(8-1)3=23$ miles, distance B travels the last day.

$S=\left(\frac{23+2}{2}\right)8=100$ miles, whole distance B travels.

As the island is 50 miles in circumference, B has been round it twice; and A has been round it once and 30 miles more: $80-50=30$ miles; $50-30=20$ miles.

Hence they will be 20 miles apart on the 8th day.

Ex. (21.) As Smith travels 15 miles per day, and travels 20 days, it is evident he will travel $15\times20=300$ miles, and that he will be $440-300=140$ miles from Washington.

As Jones travels 2 miles less on each day than the preceding, he will on the last day travel $L=25+(15.-1)-2=-3$ miles, and the whole distance he will travel will be

$$S=\left(\frac{L+a}{2}\right)n=\left(\frac{-3+25}{2}\right)15=165 \text{ miles.}$$

He will also be $440-165=275$ miles from Washington.

They will therefore be $275-140=135$ miles apart.

Ex. (22.) $L=a+(n-1)-d=\frac{1}{2}+(20-1)-\frac{1}{6}=\ 2\frac{2}{3}$.

$$S=\left(\frac{L+a}{2}\right)n=\left(\frac{-2\frac{2}{3}+\frac{1}{2}}{2}\right)20=-21\frac{2}{3}.$$

Ex. (23.) By transposing the formula, $d=\frac{2S-2an}{n(n-1)}$, we find the value of n to be

$$n=\frac{\sqrt{(2a-d)^2+8dS}-2a+d}{2d}=$$

$$n=\frac{\sqrt{(2\times\frac{1}{3}-(-\frac{1}{12}))^2+8\times-\frac{1}{12}\times-1\frac{1}{2}}-2\times\frac{1}{3}+(-\frac{1}{12})}{2\times(-\frac{1}{12})}=12.$$

Ex. (24.) $n=\frac{2S}{L+a}=\frac{2\times 37}{2\frac{1}{2}+\frac{7}{12}}=24.$

Ex. (25.) $d=\frac{L-a}{n-1}=\frac{17-3}{29-1}=\frac{1}{2}$, the common difference.

$3+\frac{1}{2}=3\frac{1}{2}$; $3\frac{1}{2}+\frac{1}{2}=4$; $4+\frac{1}{2}=4\frac{1}{2}$; $4\frac{1}{2}+\frac{1}{2}=5$; $5+\frac{1}{2}=5\frac{1}{2}$.

Hence the series, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, &c.

Ex. (26.) $a=\frac{S}{n}-\frac{(n-1)d}{2}=\frac{16\frac{1}{4}}{10}-\frac{(10-1)\frac{1}{4}}{2}=\frac{1}{2}$, the common difference.

Ex. (27.) $L=a+(n-1)d=-5+(9-1)1\frac{1}{2}=7$, the 9th term.

Ex. (28.) $15-(-1)=16$; $16\div\overline{3+1}=4$, common difference.

$4+(-1)=3$; $3+4=7$; $7+4=11$.

Hence the three means are 3, 7 and 11.

Ex. (29.) $d=\frac{2S-2an}{n(n-1)}=\frac{\overline{2\times 6\frac{7}{8}}-\overline{2\times 1\frac{1}{4}\times 10}}{10(10-1)}=-\frac{1}{8}.$

Ex. (30.) Let $\overline{x-y}$, x, and $\overline{x+y}$, be the three terms.

Then $\overline{x-y}+x+\overline{x+y}=10$.

$3x=10$.

$x=3\frac{1}{3}$, the second term.

$33\frac{1}{3}\div 3\frac{1}{3}=10$, third term.

$10-3\frac{1}{3}=6\frac{2}{3}$, common difference.

$3\frac{1}{3}-6\frac{2}{3}=-3\frac{1}{3}$, first term.

Hence the terms are $-3\frac{1}{3}$, $3\frac{1}{3}$ and 10.

Ex. (31.) Let x = the first term; then, by the question, $4x$ = the last term, and $\frac{3x^2}{4}$ = the sum of the series. The number of terms will be

$$n=\frac{2S}{L+a}=\frac{2\times\frac{3x^2}{4}}{4x+x}=\frac{3x}{10}\text{ number of terms.}$$

And, by the question, $d=2\times\frac{3x}{10}=\frac{3x}{5}$, the common difference.

Then, if we assume $x=20$, the other terms of the progression will be 32, 44, 56, 68, 80, and the sum of the series will be 300.

But, if we assume $x=40$, the other terms will be 64, 88, 112, 136, 160, and the sum of the series 1200.

An indefinite number of answers may be obtained for this question by giving any value to x.

Ex. (32.) Let $\overline{x-3y}$, $\overline{x-y}$, $\overline{x+y}$, $\overline{x+3y}$, be the numbers.

Then, $\overline{x-3y}+\overline{x-y}+\overline{x+y}+\overline{x+3y}=28$.

Collecting terms, $4x=28$.

Dividing, $x=7$.

Expanding the terms,

$$\overline{x-3y}^2=x^2-6xy+9y^2.$$
$$\overline{x-y}^2=x^2-2xy+y^2.$$
$$\overline{x+y}^2=x^2+2xy+y^2.$$
$$\overline{x+3y}^2=x^2+6xy+9y^2.$$

Sum of the squares, $4x^2+20y^2=216$.

Value of $4x^2=196$, $196+20y^2=216$.

Transposing, $20y^2=20$.

Dividing, $y^2=1$.

Evolving, $y=\pm1$.

$x-3y=7-3=4$; $x-y=7-1=6$; $x+y=7+1=8$; $x+3y=7+3=10$.

Hence the four numbers are 4, 6, 8, 10.

Ex. (33.) Let $x-y$, x, $x+y$, be the numbers.

1. Then $\overline{x-y}+x+\overline{x+y}=9$; $3x=9$: $x=3$.

2. First term evolved, $\overline{x-y}^3=x^3-3x^2y+3xy^2-y^3$.
3. Second term evolved, $x^3=x^3$.
4. Third term evolved, $\overline{x+y}^3=x^3+3x^2y+3xy^2+y^3$.
5. Sum of the cubes, $3x^3+6xy^2=99$.
6. Dividing, $x^3+2xy^2=33$.
7. Substituting the value of x^3, &c., $27+6y^2=33$.
8. Transposing, $6y^2=6$.
9. Dividing, $y^2=1$.
10. Evolving, $y=\pm1$.

$$x-y=3-1=2;\ x=3;\ x+y=3+1=4.$$

Hence the numbers are 2, 3, 4.

Ex. (34.) Let $\overline{x-3y}$, $\overline{x-y}$, $\overline{x+y}$, $\overline{x+3y}$, be the four terms.

1. Then by first condition,

$$\overline{x-3y}^2+\overline{x-y}^2=2x^2-8xy+10y^2=34.$$

2. By second condition,

$$\overline{x+y}^2+\overline{x+3y}^2=2x^2+8xy+10y^2=130.$$

3. Adding (1) and (2), $4x^2+20y^2=164$.
4. Dividing by 4, $x^2+5y^2=41$.
5. Subtracting (1) from (2), $16xy=96$.
6. Dividing, $xy=6$.
7. Putting $\left(\frac{6}{y}\right)^2$ for its equal x^2, $\frac{36}{y^2}+5y^2=41$.
8. Clearing of fractions, &c., $5y^4-41y^2=-36$.
9. Dividing, $y^4-\frac{41y^2}{5}=-\frac{36}{5}$.
10. Completing the square,

$$y^4-\frac{41y^2}{5}+\frac{1681}{100}=-\frac{36}{5}+\frac{1681}{100}=\frac{961}{100}.$$

11. Evolution, $y^2-\frac{41}{10}=\pm\frac{31}{10}$.
12. Transposing, $y^2=\pm\frac{31}{10}+\frac{41}{10}=\frac{10}{10}=1$.
13. Evolving, $y=1$.
14. $xy=1\times x=6$. Therefore, $x=6$.
15. Hence, $6-3=3$; $6-1=5$; $6+1=7$; $6+3=9$.
16. Therefore the numbers are 3, 5, 7, 9.

Ex. (35.) Let $x-y$, x, $x+y$ = the three digits.

1. Then, by first condition, $\frac{100(x-y)+10x+x+y}{3x}=27\frac{4}{7}$.
2. Multiplying and transposing, $111x-99y=82\frac{5}{7}x=\frac{579x}{7}$.
3. Collecting terms, $\frac{198x}{7}=99y$.
4. Clearing of fractions, $198x=693y$.

And, we find by the second condition of the question, that

5. $100(x-y)+10x+x+y+396=100(x+y)+10x+x-y$.
6. Multiplying and transposing, $-99y+396=99y$.
7. Collecting terms, $198y=396$.
8. Dividing, $y=2$.
9. We find (4) that $198x=693y$.
10. Therefore, $198x=2\times693$.
11. Dividing, $x=7$.
12. Hence, $x-y=7-2=5$; $x=7$; $x+y=7+2=9$.
13. Therefore the number is 579.

Ex. (36.) Let $x-3y$, $x-y$, $x+y$, $x+3y$, be the numbers.

1. Then, $\overline{x-3y}^2+\overline{x+3y}^2=2x^2+18y^2=90$.
2. And $\overline{x-y}^2+\overline{x+y}^2=2x^2+2y^2=74$.
3. Subtracting (2) from (1), $16y^2=16$.
4. Dividing, $y^2=1$.
5. Evolving, $y=\pm1$.
6. Putting the value of y into (2), $2x^2+2=74$.
7. Transposing and dividing, $x^2=36$.
8. Evolving, $x=6$.

Hence $x-3y=6-3=3$; $x-y=6-1=5$; $x+y=6+1=7$; $x+3y=6+3=9$.

Therefore the four numbers are 3, 5, 7, 9.

Ex. (37.) Let $x-3y$, $x-y$, $x+y$, $x+3y$, be the numbers.

1. Then, by first condition,

$$\overline{x-3y}+\overline{x-y}+\overline{x+y}+x+3y=14.$$

2. Collecting terms, $4x=14$.
3. Dividing, $x=3\frac{1}{2}=\frac{7}{2}$.

4. Product of first and fourth terms,

$$\overline{x-3y}\times\overline{x+3y}=x^2-9y^2.$$

5. Product of third and second terms,

$$\overline{x-y}\times\overline{x+y}=x^2-y^2.$$

6. Product of all the terms,

$$\overline{x^2-9y^2}\times\overline{x^2-y^2}=x^4-10x^2y^2+9y^4.$$

7. By second condition, &c.,

$$9y^4-10x^2y^2+x^4=120.$$

8. By substitution,

$$9y^4-\frac{490y^2}{4}+\frac{2401}{16}=120.$$

9. Dividing, $y^4-\frac{490y^2}{36}+\frac{2401}{144}=\frac{40}{3}.$

10. Transposing, &c., $y^4-\frac{245y^2}{18}=\frac{40}{3}-\frac{2401}{144}=-\frac{481}{144}.$

11. Completing the square,

$$y^4-\frac{245y^2}{18}+\frac{60025}{1296}=-\frac{481}{144}+\frac{60025}{1296}=\frac{55696}{1296}.$$

12. Evolving, $y^2-\frac{245}{36}=\pm\frac{236}{36}.$

13. Transposing, $y^2=\pm\frac{236}{36}+\frac{245}{36}=\frac{481}{36}$, or $\frac{9}{36}.$

But the conditions of the question will only admit of the last value of y^2.

Therefore, $y^2=\frac{9}{36}.$

Evolving, $y=\frac{3}{6}=\frac{1}{2}.$

Hence $3\frac{1}{2}-1\frac{1}{2}=2$; $3\frac{1}{2}-\frac{1}{2}=3$; $3\frac{1}{2}+\frac{1}{2}=4$; $3\frac{1}{2}+1\frac{1}{2}=5$.
We therefore find the numbers to be 2, 3, 4, 5.

Ex. (38.) Let the less number be represented by y, and the common difference by x; the four required numbers will then be expressed by y, $y+x$, $y+2x$, $y+3x$.

Therefore, by the question, we have the two following equations.

1. $y\times\overline{y+3x}=y^2+3xy=112.$

2. And $\overline{y+x}\times\overline{y+2x}=y^2+3xy+2x^2=120.$

3. Subtracting the first of these equations from the last, we have $2x^2=8$; $x^2=4$; $x=2$.

But, to find y, we have given,

4. Supplying the value of x in (1), $y^2+6y=112$.
5. Completing the square, $y^2+6y+9=112+9=121$.
6. Evolving, $y+3=11$.
7. Transposing, $y=11-3=8$.
8. And $8+2=10$; $10+2=12$; $12+2=14$.

Hence the four numbers are 8, 10, 12, 14.

Ex. (39.) Let x = the number of days required.

1. Then, $\frac{\overline{1+x}\times x}{2}=\frac{x^2+x}{2}$, number of miles A travelled.
2. And $\frac{42x-2x^2}{2}$ = the number of miles B travelled.
3. Then, $\frac{x^2+x}{2}+\frac{42x-2x^2}{2}=165$.
4. And $x+x^2+42x-2x^2=330$.
5. Transposing, &c., $x^2-43x=-330$.
6. Completing the square,

$$x^2-43x+\frac{1849}{4}=-330+\frac{1849}{4}=\frac{529}{4}.$$

7. Evolving, $x-\frac{43}{2}=\pm\frac{23}{2}$.
8. Transposing, $x=\pm\frac{23}{2}+\frac{43}{2}=10$ days, or 33 days.

Hence it appears that they meet in 10 days.

On the tenth day B travels two miles, and the next day he rests; the following day he returns two miles, the succeeding day four miles, and so on, increasing two miles every day; and, on the thirty-third day, he again comes up with A, who has been travelling forward, every day's journey being one mile longer than that of the preceding day.

Ex. (40.) Let $x+6, x+2, x-2, x-6$, represent the numbers.

1. Then,

$$(x+6)(x+2)(x-2)(x-6)=x^4-40x^2+144=1680.$$

2. Adding 256 to both sides of the equation, we have

$$x^4-40x^2+400=1936.$$

3. Evolving, $x^2-20=\pm 44.$
4. Transposing, $x^2=\pm 44+20=64$, or -24.
5. $x=\pm 8$, or $\pm 2\sqrt{-6}.$
6. $8+6=14,\ 8+2=10,\ 8-2=6,\ 8-6=2.$
7. The numbers therefore are 14, 10, 6, 2.
8. The other value of x is impossible.

Ex. (41.) Let $x-2y$, $x-y$, x, $x+y$, $x+2y$ = the number of days.

Then, $\overline{x-2y}+\overline{x-y}+x+\overline{x+y}+\overline{x+2y}=5x=20$, $x=4$.

As the first person could reap one acre in $x-2y$ days, it is evident he would in one day reap $\frac{1}{x-2y}$ part of an acre; and the second person $\frac{1}{x-y}$ part of an acre; and the third person $\frac{1}{x}=\frac{1}{4}$ part; the fourth person $\frac{1}{x+y}$ part; and the fifth person $\frac{1}{x+2y}$ part of an acre. Therefore they all in one day would reap $\frac{1}{x-2y}+\frac{1}{x-y}+\frac{1}{x+y}+\frac{1}{x+2y}=\frac{87}{60}-\frac{1}{4}=\frac{6}{5}$ of an acre.

We find their sum to be,

$$\left.\begin{array}{r}(x^2-y^2).(x+2y)\\ +(x^2-4y^2).(x+y)\\ +(x^2-4y^2).(x-y)\\ +(x^2-y^2).(x-2y)\end{array}\right\}\text{Numerators.}$$

$$(x^2-y^2).(x^2-4y^2),\ \text{common denominator.}$$

By adding the numerators, we have

$$\frac{2x(x^2-y^2)+2x(x^2-4y^2)}{(x^2-y^2).(x^2-4y^2)}=\frac{6}{5}.$$

Therefore, $2x(x^2-y^2)+2x(x^2-4y^2)=\frac{6}{5}(x^2-y^2).(x^2-4y^2).$

Adding and multiplying, $2x(2x^2-5y^2)=\frac{6}{5}(x^4-5x^2y^2+4y^4).$

Putting the value of $x=4$, as found above, in this equation, we have $128-20y^2=\frac{3}{5}(256-80y^2+4y^4).$

Multiplying, $640-100y^2=768-240y^2+12y^4.$

Dividing, $160-25y^2=192-60y^2+3y^4.$

Transposing, $y^4-\frac{35y^2}{3}=-\frac{32}{3}.$

Completing the square,

$$y^4-\frac{35y^2}{3}+\frac{1225}{36}=-\frac{32}{3}+\frac{1225}{36}=\frac{841}{36}.$$

Evolving, $y^2-\frac{35}{6}=\pm\frac{29}{6}.$

Transposing, $y^2=\pm\frac{29}{6}+\frac{35}{6}=-\frac{6}{6}=1.$

Evolving, $y=\pm1.$

Hence the value of $x=4$, and $y=\pm1$.

Therefore the numbers are 2, 3, 4, 5, 6 days.

Ex. (42.)

$$S=\tfrac{1}{2}n[2a+(n-1)-d]=\tfrac{11}{2}[\overline{2\times25}+(11-1)-5]=0.$$

By the above process we find he has travelled back to Boston by the end of the eleventh day.

Ex. (43.) We find he will have to travel forty miles and two rods, = 12802 rods, to bring the most distant stone, and two rods for the nearest stone, and that the number of stones will be one more than the number of rods in forty miles; therefore we have the following formula for the solution of the problem

$$S=\left(\frac{L+a}{2}\right)n=\left(\frac{12802+2}{2}\right)6401=128{,}060 \text{ miles, 2 rods.}$$

GEOMETRICAL PROGRESSION.

Art. 274. (p. 234.)

Ex. (7.) $r=\left(\frac{L}{a}\right)^{\frac{1}{n-1}}=\left(\frac{64}{\frac{1}{16}}\right)^{\frac{1}{5}}=(1024)^{\frac{1}{5}}=4$, ratio.

Ex. (8.) $a=\left(\frac{L}{r^{n-1}}\right)=\frac{135}{3^3}=\frac{135}{27}=5$, first term.

ART. 276. (p. 235.)

Ex. (11.) $r=\left(\frac{L}{a}\right)^{\frac{1}{m+1}}=\left(\frac{64}{1}\right)^{\frac{1}{6}}=\frac{2}{1}=2$, ratio.

$1\times2=2$; $2\times2=4$; $4\times2=8$; $8\times2=16$; $16\times2=32$.
The means, therefore, are 2, 4, 8, 16, 32.

Ex. (12.) $r=\left(\frac{L}{a}\right)^{\frac{1}{m+1}}=\left(\frac{288}{18}\right)^{\frac{1}{2}}=\left(\frac{144}{9}\right)^{\frac{1}{2}}=\frac{12}{3}=4$, ratio.
$18\times4=72$ rods.

ART. 279. (p. 237.)

Ex. (15.) $S=\frac{ar^n-a}{r-1}=\frac{144\times1.06^4-144}{1.06-1}=629.944704$, sum of the series.

Ex. (16.) $S=\frac{a-ar^n}{1-r}=\frac{(9-(9\times(\frac{1}{4})^6)}{1-\frac{1}{4}}=\frac{\frac{9}{1}-\frac{9}{4096}}{\frac{3}{4}}=\frac{36855}{4096}\times\frac{4}{3}$ $=11\frac{1021}{1024}$, sum of the series.

Ex. (18.) $S=\frac{ar^n-a}{r-1}=\frac{(1\times2^7)-1}{2-1}=127$, sum of the series.

Ex. (19.) $S=\frac{ar^n-a}{r-1}=\frac{(5\times10^7)-5}{10-1}=5555555$, sum of the series.

Ex. (20.) $S=\frac{a-ar^n}{1-r}=\frac{4-4\times(\frac{1}{4})^5}{1-\frac{1}{4}}=\frac{\frac{1023}{256}}{\frac{3}{4}}=5\frac{21}{64}$, sum of the series.

Ex. (21.) $S=\frac{a-ar^n}{1-r}=\frac{5-5\times(\frac{1}{5})^5}{1-\frac{1}{5}}=\frac{3124}{625}\times\frac{5}{4}=6\frac{31}{125}$, sum of the series.

Ex. (22.) $S=\frac{ar^n-a}{r-1}=\frac{3\times3^9-3}{3-1}=\frac{59046}{2}=\$295.23$.

Ex. (23.) $L=\frac{a+(r-1)S}{r}=\frac{3+(2-1)93}{2}=48$, last term.

Ex. (24.) $\frac{128}{\frac{1}{2}}=256$; $\sqrt[4]{256}=4$.
$\frac{1}{2}\times4=2$; $2\times4=8$; $8\times4=32$.
The means are, therefore, 2, 8, 32.

Ex. (25.) $r=\sqrt[n-1]{\frac{L}{a}}=\sqrt[7]{\frac{4374}{2}}=\sqrt[7]{2187}=3$, ratio.

Ex. (26.) $a=\frac{L}{r^{n-1}}=\frac{128}{2^5}=\frac{128}{32}=4$, first term.

Ex. (27.) $L=ar^{n-1}=\frac{10}{3}\left(\frac{3}{5}\right)^7=\frac{21870}{234375}=\frac{1458}{15625}$, last term

$$S=\frac{a-ar^n}{1-r}=\frac{\frac{10}{3}-\frac{10}{3}(\frac{3}{5})^8}{1-\frac{3}{5}}=\frac{57609600}{7031250}=8\tfrac{9064}{46875}.$$

Ex. (28.) $r=\sqrt[n-1]{\frac{L}{a}}=\sqrt[6]{\frac{64}{1}}=\sqrt[6]{64=2}$, ratio.

$$S=\frac{Lr-a}{r-1}=\frac{\overline{64\times2-1}}{2-1}=127, \text{ sum of the series.}$$

Ex. (29.) $(127-64)=63$; $127\div63=2$, ratio, and the remainder is the first term.

Ex. (30.) $L=ar^{n-1}=2\times4^{11}=8388608$, last term.

$$S=\frac{ar^n-a}{r-1}=\frac{2\times4^{12}-2}{4-1}=11184810, \text{ sum of the series.}$$

Ex. (31.) Let $\frac{x}{y}$, x, xy, be the numbers.

1. First condition, $\frac{x}{y}\times x\times xy=x^3=64$, $x=4$.
2. Second condition, $\frac{x^3}{y^3}+x^3+x^3y^3=584$.
3. Dividing by x^3, $\frac{1}{y^3}+1+y^3=\frac{584}{64}=\frac{73}{8}$.
4. Transposing, $\frac{1}{y^3}+y^3=\frac{65}{8}$.
5. Multiplying by y^3, $1+y^6=\frac{65y^3}{8}$.
6. Transposing, $y^6-\frac{65y^3}{8}=-1$.
7. Completing the square,

$$y^6-\frac{65y^3}{8}+\frac{4225}{256}=-1+\frac{4225}{256}=\frac{3969}{256}.$$

8. Evolving, $y^3-\frac{65}{16}=\pm\frac{63}{16}$.

9. Transposing, $y^3=\pm\frac{63}{16}+\frac{65}{16}=8$, or $\frac{1}{8}$.

10. Evolving, $y=2$, or $\frac{1}{2}$.

Hence the numbers are 2, 4, 8.

The second value of y will not answer the conditions of the question.

Ex. (32.) Let x, xy, xy^2, xy^3, be the numbers.

1. Then, by the question, $x+xy^3 : xy+xy^2 :: 7 : 3$.
2. Dividing by $x+xy$, $1-y+y^2 : y :: 7 : 3$.
3. By Prop. VIII. (p. 209), $1+y^2 : y :: 10 : 3$.
4. Multiplying extremes, &c., $3y^2+3=10y$.
5. Dividing, $y^2-\frac{10y}{3}=-1$.
6. Completing the square,

$$y^2-\frac{10y}{3}+\frac{25}{9}=-1+\frac{25}{9}=\frac{16}{9}.$$

7. Evolving, $y-\frac{5}{3}=\pm\frac{4}{3}$.
8. Transposing, $y=\pm\frac{4}{3}+\frac{5}{3}=3$, or $\frac{1}{3}$.
9. By the question, &c., $xy^3-xy=27x-3x=24$.
10. And $24x=24$.
11. Dividing, $x=1$.

$$1;\ 1\times3=3;\ 3\times3=9;\ 9\times3=27.$$

The numbers therefore are 1, 3, 9, 27.

Ex. (33.) Let the series be $\frac{x}{y^2}$, $\frac{x}{y}$, x, xy.

1. By second condition, $xy-\frac{x}{y^2}=49$.
2. By first condition, $x-\frac{x}{y}=14$.
3. Clearing (1) of fractions, $xy^3-x=49y^2$.

4. Clearing (2) of fractions, $xy-x=14y.$
5. Dividing (4) into factors, $x(y-1)=14y.$
6. Dividing (3) into factors, $x(y^3-1)=49y^2.$
7. Dividing (3) by (2), $y^3+y^2+y=\frac{49y^2}{14}.$
8. Transposing and reducing, $y^2-\frac{35y}{14}=-1.$
9. Completing the square,

$$y^2-\frac{35y}{14}+\frac{1225}{784}=-1+\frac{1225}{784}=\frac{441}{784}.$$

10. Evolving, $y-\frac{35}{28}=\frac{21}{28}.$
11. Transposing, $y=\frac{21}{28}+\frac{35}{28}=\frac{56}{28}=2.$
12. Putting the value of y into (2), $x-\frac{x}{2}=14.$
13. Clearing of fractions, $2x-x=28.$
14. Cancelling, $x=28.$
15. Putting the values of x and y into the first term,

$$\frac{x}{y^2}=\frac{28}{4}=7.$$

16. Putting the values of x and y into the second term,

$$\frac{x}{y}=\frac{28}{2}=14.$$

17. Putting the values of x and y into the fourth term,

$$xy=28\times2=56.$$

Hence the required numbers are 7, 14, 28, 56.

INFINITE SERIES.

Art. 285. (p. 243.)

Ex. (1.) Let S = the sum of an infinite series.
a = the first term.
r = the ratio.

And $S=\frac{a}{1-r}=$ the formula for obtaining the sum of the series.

Ex. (2.) $S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=2$, Ans.

Ex. (3.) $S=\frac{a}{1-r}=\frac{8}{1-\frac{1}{5}}=\frac{8}{\frac{4}{5}}=10$, Ans.

Ex. (4.) $S=\frac{a}{1-r}=\frac{\frac{2}{3}}{1-\frac{1}{2}}=\frac{\frac{2}{3}}{\frac{1}{2}}=\frac{2}{3}\times\frac{2}{1}=\frac{4}{3}=1\frac{1}{3}$, Ans.

Ex. (5.) $S=\frac{a}{1-r}=\frac{4}{1-\frac{1}{4}}=\frac{4}{\frac{3}{4}}=\frac{4}{1}\times\frac{4}{3}=\frac{16}{3}=5\frac{1}{3}$, Ans.

Ex. (6.) $S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{10}}=\frac{1}{\frac{9}{10}}=\frac{1}{1}\times\frac{10}{9}=\frac{10}{9}=1\frac{1}{9}$, Ans.

Ex. (9.) $S=\frac{a}{1-r}=\frac{\frac{571428}{1000000}}{1-\frac{1}{1000000}}=\frac{571428}{999999}=\frac{4}{7}$, Ans.

Ex. (10.) $S=\frac{a}{1-r}=\frac{\frac{857142}{1000000}}{1-\frac{1}{1000000}}=\frac{857142}{999999}=\frac{6}{7}$, Ans.

Ex. (12.) $.13\dot{8}=\frac{13}{100}+\frac{\frac{8}{1000}}{1-\frac{1}{10}}=\frac{13}{100}+\frac{8}{900}=\frac{5}{36}$, Ans.

Ex. (13.) By transposing $S=\frac{a}{1-r}$, we find $r=1-\frac{a}{S}$; therefore $r=1-\frac{a}{S}=1-\frac{8}{10}=\frac{1}{5}$, Ans.

Ex. (14.) $r=1-\frac{a}{S}=1-\frac{\frac{2}{3}}{1\frac{1}{3}}=1-\overline{\frac{2}{3}\times\frac{3}{4}}=\frac{1}{2}$, Ans.

Ex. (15.) By multiplying $S=\frac{a}{1-r}$ by $1-r$, we have $a=S-Sr$, therefore $a=10-(10\times\frac{1}{5})=8$, Ans.

SIMPLE INTEREST.

ART. 287. (p. 245.)

Ex. (2.) $i=ptr=380\times.05\times10=\$190.$

Ex. (3.) $i=ptr=890.75\times.08\times35=\$249.41.$

Ex. (4.) $i=ptr=17.18\times.045\times5.19\frac{4}{9}=\$4.02.$

Ex. (6.) $a=p+ptr=800+\overline{800\times.06\times6.1\frac{1}{6}}=\$1093.60.$

Ex. (7.) $a=p+ptr=670.18+670.18\times.09\times3.6\frac{7}{18}=$ $\$889.66.$

ART. 288. (p. 246.)

Ex. (9.) $p=\frac{a}{1+tr}=\frac{570}{1+(10\times.05)}=\$380.$

Ex. (10.) $p=\frac{a}{1+tr}=\frac{1140.16}{1+(3.5\times.08)}=\$890.75.$

Ex. (11.) $p=\frac{a}{1+tr}=\frac{21.20}{1+(5.19\frac{4}{9}\times.045)}=\$17.18.$

Ex. (12.) $p=\frac{a}{1+tr}=\frac{178.56}{1+(3\times.08)}=\$144.$

Ex. (13.) $p=\frac{a}{1+tr}=\frac{889.66}{1+(3.6\frac{7}{18}\times.09)}=\$670.18.$

ART. 289. (p. 246.)

Ex. (15.) $t=\frac{a-p}{rp}=\frac{570-380}{.05\times380}=\frac{190}{19}=10$ years.

Ex. (16.) $t=\frac{a-p}{rp}=\frac{1140.16-890.75}{.08\times890.75}=3.5=3$ years, 6 mos.

Ex. (17.) $t=\frac{a-p}{rp}=\frac{21.20-17.18}{.045\times17.18}=5.19\frac{4}{9}=5$ years, 2 mos., 10 days.

Ex. (18.) $t=\frac{a-p}{rp}=\frac{889.66-670.18}{.09\times670.18}=3.638\frac{2}{3}=3$ years, 7 months, 20 days.

Ex. (19.) $t=\frac{a-p}{rp}=\frac{353.60-320}{.09\times320}=1.16\frac{2}{3}=1$ year, 2 months, March 1, 1852.

ART. 290. (p. 247.)

By transposing the last formula, we obtain the following for finding r, the rate per cent.

Ex. (20.) $r=\frac{a-p}{pt}=\frac{570-380}{380\times10}=.05$ per cent.

Ex. (21.) $r=\frac{a-p}{pt}=\frac{1140.16-890.75}{890.75\times3.5}=.08$ per cent.

Ex. (22.) $r=\frac{a-p}{pt}=\frac{21.20-17.18}{17.18\times5.194\frac{4}{9}}=.045$ per cent.

Ex. (23.) $r=\frac{a-p}{pt}=\frac{219.48}{670.18\times3.638\frac{2}{3}}=.09$ per cent.

Ex. (24.) $r=\frac{a-p}{pt}=\frac{153.648-144}{144\times3.35}=.02$ per cent.

Ex. (25.) $60\times.95=57$ inches; $57=1.583\frac{1}{3}$ yards.
$\frac{7.5}{1.583\frac{1}{3}}=\frac{22.5}{4.75}$; $\frac{22.5}{4.75}\times\frac{100}{96}=\frac{2250}{456}=4$ yards, $33\frac{12}{19}$ inches.

DISCOUNT AT SIMPLE INTEREST.

ART. 291. (p. 249.)

Ex. (2.) $p=\frac{S}{1+tr}=\frac{372}{1+(4\times.06)}=\$300.$

Ex. (3.) $p=\frac{S}{1+tr}=\frac{133.20}{1+(1\frac{2}{3}\times.0825)}=\$117.09.$

Ex. (4.) $p=\frac{S}{1+tr}=\frac{21.20}{1+(5.19\frac{4}{7}\times.045)}=\$17.18.$
$\$21.20-17.18=\$4.02.$

Ex. (5.) $p=\frac{S}{1+tr}=\frac{353.60}{1+(1\frac{1}{6}\times .09)}=\$320.$

Ex. (6.) $p=\frac{S}{1+tr}=\frac{144}{1+(10\times .06)}=\frac{176.40}{1+(8\times .12)}=\$0.00.$

Hence they are both of the same value.

Ex. (7.) $p=\frac{S}{1+tr}=\frac{1725}{1+(1\times .15)}=\$1500.$

Ex. (8.) $r=\frac{S-p}{tp}=\frac{162-150}{.5\times 150}=\frac{12}{75}=.16$ per cent.

Ex. (9.) $t=\frac{S-p}{rp}=\frac{590-500}{.06\times 500}=\frac{90}{30}=3$ years.

Ex. (10.) $p=\frac{S}{1+tr}=\frac{200}{1+(\frac{1}{6}\times .15)}=\frac{200}{1.025}=\$195.12+$

Ex. (11.) $p=\frac{S}{1+tr}=\frac{1827}{1+(100\times .06)}=\$261.$

PARTNERSHIP, OR COMPANY BUSINESS.

ART. 293. (p. 253.)

Ex. (3.) Let m = the sum M put in,
And n = the sum N put in, and

Let m and n represent M and N's stock, and a the sum gained, and x and y the sum each gained.

Then $x=\frac{am}{m+n}=\frac{120\times 500}{500+750}=\$48,$ M's gain.

And $y=\frac{an}{m+n}=\frac{120\times 750}{500+750}=\$72,$ N's gain.

Ex. (4.) Let x and y = the sums Q and X each respectively paid for the use of the pasture; and m = Q's cows, and n = X's cows, and a = the price paid for the use of the pasture.

$$\text{Then, } x=\frac{am}{m+n}=\frac{120\times11}{11+15}=\$50.76\tfrac{12}{13},\ \text{Q pays.}$$

$$\text{And }\quad y=\frac{an}{m+n}=\frac{120\times15}{11+15}=\$69.23\tfrac{1}{13},\ \text{X pays.}$$

Ex. (5.) Let m and n represent the sum A and B paid respectively; a = the sum they gained; and x and y = the sum each respectively should receive.

$$\text{Then, } x=\frac{am}{m+n}=\frac{1500\times10000}{10000+7000}=\$882\tfrac{6}{17},\ \text{A's gain.}$$

$$\text{And }\quad y=\frac{an}{m+n}=\frac{1500\times7000}{10000+7000}=\$617\tfrac{11}{17},\ \text{B's gain.}$$

Ex. (6.) Let m, n and p, represent A, B and C's share of the gain, and a = their capital. And let x, y, z = the stock of each man respectively.

$$\text{Then, } x=\frac{ma}{m+n+p}=\frac{100\times6000}{100+80+60}=\$2500,\ \text{A's stock.}$$

$$\text{“}\quad y=\frac{na}{m+n+p}=\frac{80\times6000}{100+80+60}=\$2000,\ \text{B's stock.}$$

$$\text{And }\quad z=\frac{pa}{m+n+p}=\frac{60\times6000}{100+80+60}=\$1500,\ \text{C's stock.}$$

Ex. (7.) We first reduce the horses and cows to oxen; thus,

As 2 horses : 5 horses : : 3 oxen : $7\frac{1}{2}$ oxen. A's
5 cows : 9 cows : : 4 oxen : $7\frac{1}{5}$ oxen. C's.
A's $7\frac{1}{2}$, + B's 7, + C's $7\frac{1}{5}$ oxen, = $21\frac{7}{10}$ oxen.

Let m, n, p, respectively represent A, B and C's oxen; a = the sum paid for their pasture. And let x, y, z = the part of the expenses each pays respectively.

$$\text{Then, } x=\frac{ma}{m+n+p}=\frac{7\frac{1}{2}\times100}{7\frac{1}{2}+7+7\frac{1}{5}}=\$34.56\tfrac{48}{217},\ \text{A pays.}$$

$$\text{“}\quad y=\frac{na}{m+n+p}=\frac{7\times100}{7\frac{1}{2}+7+7\frac{1}{5}}=\$32.25\tfrac{175}{217},\ \text{B pays.}$$

$$\text{And }\quad z=\frac{pa}{m+n+p}=\frac{7\frac{1}{5}\times100}{7\frac{1}{2}+7+7\frac{1}{5}}=\$33.17\tfrac{211}{217},\ \text{C pays.}$$

Ex. (8.) The field contained $40\times40=1600$ square rods.

A reaped $25\times25=625$ square rods.

B reaped 400 square rods.

And C reaped $1600-(625+400)=575$ square rods.

Let m, n, p, represent the square rods A, B and C reaped respectively, and a = the sum paid for reaping. Let x, y, z = the sum each received respectively.

Then, $x=\frac{ma}{m+n+p}=\frac{625\times32}{625+400+575}=\$12.50$, A received.

" $y=\frac{na}{m+n+p}=\frac{400\times32}{625+400+575}=\$8$, B received.

And $z=\frac{pa}{m+n+p}=\frac{575\times32}{625+400+575}=\$11.50$, C received.

PARTNERSHIP ON TIME.

ART. 293. (p. 255.)

Ex. (10.) Let m, n, represent each man's stock respectively; a = the whole gain; and t, t', the time each man's stock was in trade.

Let x and y = A's and B's gain respectively.

Then, $x=\frac{mta}{mt+nt'}=\frac{3000\times12\times340}{(3000\times12)+(4000\times8)}=\$180$, A's gain.

And $y=\frac{nt'a}{mt+nt'}=\frac{4000\times8\times340}{(3000\times12)+(4000\times8)}=\$160$, B's gain.

Ex. (11.) Let m, n, p, represent each man's stock; a = the whole gain. Let x, y, z = A, B and C's gain respectively; and t, t', t'', the time in trade.

Then, $x=\frac{mta}{mt+nt'+pt''}=\frac{300\times10\times120}{(300\times10)+(400\times8)+(600\times2)}=$ $\$48.64\frac{32}{37}$, A's gain.

Then, $y=\frac{nt'a}{mt+nt'+pt''}=\frac{400\times8\times120}{(300\times10)+(400\times8)+(600\times2)}=$ $\$51.89\frac{7}{37}$, B's gain.

And $z=\frac{pt''a}{mt+nt'+pt''}=\frac{600\times2\times120}{(300\times10)+(400\times8)+(600\times2)}=$ $\$19.45\frac{35}{37}$, C's gain.

Ex. (12.) Let m, n, p, represent the oxen of A, B and C; a = the price they paid for pasturing; and t, t', t'' = the times each had his oxen in the pasture.

Let x, y, z = the sums each paid respectively.

Then, $x=\frac{mta}{mt+nt'+pt''}=\frac{24\times12\times76.80}{(24\times12)+(25\times12)+(30\times6)}=$ \$28.80, A paid.

And $y=\frac{nt'a}{mt+nt'+pt''}=\frac{25\times12\times76.80}{(24\times12)+(25\times12)+(30\times6)}=$ \$30, B paid.

And $z=\frac{pt''a}{mt+nt'+pt''}=\frac{30\times6\times76.80}{(24\times12)+(25\times12)+(30\times6)}=$ \$18, C paid.

Ex. (13.) Let t' = the time the house was occupied by Jones.

t'' = the time the house was occupied by Jones and Smith.

t''' = the time the house was occupied by Roe, Jones and Smith.

a = the rent of the house for one year.

x, y, z = the share of the rent each paid respectively.

Then, $x=\frac{t'a}{t}+\frac{\frac{1}{2}(t'a)}{t}+\frac{\frac{1}{3}(t''a)}{t}=\frac{3\times500}{12}+\frac{\frac{1}{2}(6\times500)}{12}+\frac{\frac{1}{3}(3\times500)}{12}=\$291\frac{2}{3}$, Jones' share.

Then, $y=\frac{\frac{1}{2}(t'a)}{t}+\frac{\frac{1}{3}(t''a)}{t}=\frac{\frac{1}{2}(6\times500)}{12}+\frac{\frac{1}{3}(3\times500)}{12}=\$166\frac{2}{3}$, Smith's share.

And $z=\frac{\frac{1}{3}(t''a)}{t}=\frac{\frac{1}{3}(3\times500)}{12}=\$41\frac{2}{3}$, Roe's share.

Ex. (14.) Let d = twice the distance from Boston to Worcester.

d' = the distance A and B rode before they took in C.

d'' = the distance A, B and C rode before they took in D.

d''' = the distance they rode after they took in D.

a = the sum paid for the coach.

w, x, y, z = the sum each paid respectively.

Then, $w=\frac{\frac{1}{2}(d'a)}{d}+\frac{\frac{1}{3}(d''a)}{d}+\frac{\frac{1}{4}(d'''a)}{d}=$

$\frac{\frac{1}{2}(30\times20)}{84}+\frac{\frac{1}{3}(34\times20)}{84}+\frac{\frac{1}{4}(20\times20)}{84}=\$7.46\frac{8}{252}$, A pays.

Then, $x=\frac{\frac{1}{2}(d'a)}{d}+\frac{\frac{1}{3}(d''a)}{d}+\frac{\frac{1}{4}(d'''a)}{d}=$

$\frac{\frac{1}{2}(30\times20)}{84}+\frac{\frac{1}{3}(34\times20)}{84}+\frac{\frac{1}{4}(20\times20)}{84}=\$7.46\frac{8}{252}$, B pays.

Then, $y=\frac{\frac{1}{3}(d''a)}{d}+\frac{\frac{1}{4}(d''a)}{d}=\frac{\frac{1}{3}(34\times20)}{84}+\frac{\frac{1}{4}(20\times20)}{84}=$

$\$3.88\frac{224}{252}$, C pays.

And $z=\frac{\frac{1}{4}(d''a)}{d}=\frac{\frac{1}{4}(20\times20)}{84}=\$1.19\frac{12}{252}$, D pays.

Ex. (15.) Let m represent the sum A put in, and t = the time; and n = the sum B put in, and t'' = the time; and a = their gain. Let x and y = their respective gain.

Then, $$x=\frac{mta}{mt+nt'}=\frac{abc}{ab+cd}.$$

And $$y=\frac{nt'a}{mt+nt'}=\frac{cde}{ab+cd}.$$

Ex. (16.) Let m, n, p, represent each man's gain respectively; a = the whole stock; and t, t', t'' = the times each man employed his capital.

Let x, y, z = each man's stock respectively.

Then, $\frac{m}{t}+\frac{n}{t'}+\frac{p}{t''} : \frac{m}{t} :: a : x$.

Therefore, $x=\frac{\frac{am}{t}}{\frac{m}{t}+\frac{n}{t'}+\frac{p}{t''}}=\frac{\frac{1911\times26}{3}}{\frac{26}{3}+\frac{39}{5}+\frac{52}{7}}=\$693\frac{2273}{2509}$, A's st'ck.

And $\frac{m}{t}+\frac{n}{t'}+\frac{p}{t''} : \frac{n}{t'} :: a : y$.

Therefore, $y=\frac{\frac{an}{t'}}{\frac{m}{t}+\frac{n}{t'}+\frac{p}{t''}}=\frac{\frac{1911\times39}{5}}{\frac{26}{3}+\frac{39}{5}+\frac{52}{7}}=\$623\frac{2002}{2509}$, B's st'ck.

And $\frac{m}{t}+\frac{n}{t'}+\frac{p}{t''} : \frac{p}{t'} :: a : z$.

$$\text{Therefore, } z=\frac{\frac{ap}{t''}}{\frac{m}{t}+\frac{n}{t'}+\frac{p}{t''}}=\frac{\frac{1911\times52}{7}}{\frac{26}{3}+\frac{39}{5}+\frac{52}{7}}=\$594\tfrac{234}{2509},\text{ C's st'ck.}$$

Ex. (17.) If 12 oxen eat $3\frac{1}{3}$ acres of grass in 4 weeks, and 21 oxen eat 10 acres in 9 weeks, how many acres would it require to feed 36 oxen 18 weeks, the grass to be growing uniformly?

NOTE.—In some editions of the Algebra the answer is in weeks, and in that case the answer would be *indeterminate.*

Let $x =$ the quantity of grass on each acre.

1. $12\times4\times x=48x =$ whole quantity on $3\frac{1}{3}$ acres for 4 weeks.
2. $\frac{48x}{3\frac{1}{3}} =$ whole quantity on an acre for 4 weeks.

By the second conditions of the question,

3. $21\times9\times x=189x =$ whole quantity on 10 acres for 9 weeks.
4. $\frac{189x}{10} =$ whole quantity on one acre for 9 weeks.
5. $\frac{189x}{10}-\frac{48x}{3\frac{1}{3}}=\frac{45x}{10} =$ whole quantity grown on one acre for 5 weeks.
6. $\frac{45x}{10}\div5=\frac{9x}{10} =$ quantity which grows on one acre for one week.
7. $\frac{9x}{10}\times3\frac{1}{3}\times4=12x =$ quantity which grows on $3\frac{1}{3}$ acres in 4 weeks.
8. $48x-12x=36x =$ original quantity of grass on $3\frac{1}{3}$ acres.
9. $36x\div3\frac{1}{3}=\frac{108x}{10} =$ original quantity on one acre.
10. $36\times18\times x=648x =$ the quantity of grass that 36 oxen will eat in 18 weeks.

11. $\frac{9x}{10}\times18=\frac{162x}{10}$ = the quantity of grass that grows on one acre in 18 weeks.

12. $\frac{108x}{10}+\frac{162x}{10}=27x$ = the whole quantity of grass on one acre for 18 weeks.

13. $648x\div27x=24$ acres. *Ans.*

Ex. (18.) Let a, b, c, represent the various sums A had in the firm, and t, t', t'' = the times they were invested.

Let d, e, f, represent the various sums invested by B, and u, u', u'' = the times they were continued in the firm.

Let g, h, m, p, represent the various sums C had in the firm, and, n, n', n'', n''' = the times they were invested; and let r = the sum they gained.

Let x, y, z = A, B and C's share of the gain respectively.

$$\text{Then, } x=\frac{(at+bt'+ct'')r}{at+bt'+ct''+du+eu'+fu''+gn+hn'+mn''+pn'''}=$$

$$\frac{((4\times4000)+(12\times4500)+(4\times3500)4420)}{16000+54000+14000+3000+6000+27000+12000+32000+12000+18000}=\$1680,\text{ A's gain.}$$

$$\text{Then, } y=\frac{(du+eu'+fu'')r}{at+bt'+ct''+du+eu'+fu''+gn+hn'+mn''+pn'''}=$$

$$\frac{((10\times3000)+(4\times1500)+(6\times4500)4420)}{16000+54000+14000+30000+6000+27000+12000+32000+12000+18000}=\$1260,\text{ B's.}$$

$$\text{And } z=\frac{(gn+hn'+mn''+pn''')r}{at+bt'+ct''+du+eu'+fu''+gn+hn'+mn''+pn'''}=$$

$$\frac{((6\times2000+8\times4000+2\times6000+4\times4500)4420)}{16000+54000+14000+30000+6000+27000+12000+32000+12000+18000}=\$1480,\text{ C's.}$$

INDETERMINATE ANALYSIS.

Art. 298. (p. 264.)

Ex. (8.) Here we have

$$x^2-y^2=45,\text{ or }(x-y)(x+y)=45.$$

And $$x'^2-y'^2=159,\text{ or }(x'-y')(x'+y')=159.$$

But, since $x'-y'=x-y$ by the conditions of the question, and

45 and 159 have only the common factors 1 and 3, it is evident we must have $x-y=1$, or 3; and therefore, $x+y=45$, or 15.

Whence, $x=23$, and $y=22$; or, $x=9$, and $y=6$.

NOTE. — This is evident from the fact that if we add half the difference of two numbers to half their sum, we obtain the larger; and, if we subtract half the difference of two numbers from half their sum, we obtain the less.

Also, $x'-y'=1$, or 3; and, therefore, $x'+y'=159$, or 53. Whence, $x'=80$, and $y'=79$; or, $x'=28$, and $y'=25$.

That is, at the first period their ages were 9 and 6, and at the second 28 and 25; or, at the first period they were 23 and 22, and at the second 80 and 79.

If the given differences had been *prime* to each other, there would have been only *one* solution of the problem.

EXAMPLES. (p. 264.)

Ex. (1.) Let x = the lbs. of sugar of the first kind, and y = the lbs. of the second kind.

1. Then, by conditions, $11x+5y=254$.
2. Transposing, $5y=254-11x$.
3. Dividing, $y=51-2x-\frac{1+x}{5}$.
4. Let $n=\frac{1+x}{5}$.
5. Clearing of fractions, $5n=1+x$.
6. Transposing, $x=5n-1$.
7. Substituting this last value of x for its value in (3), we have $y=51-2(5n-1)-n$.
8. Collecting terms, $y=53-11n$.

If $n=1$, then $y=42$, and $x=4$.
$n=2$, " $y=31$, " $x=9$.
$n=3$, " $y=20$, " $x=14$.
$n=4$, " $y=9$, " $x=19$.

If we were to suppose the value of $n=5$, then y would be a minus quantity, $y=53-55=-2$. We therefore find the true answers to be 19 lbs. with 9 lbs.; 14 lbs. with 20 lbs.; 9 lbs. with 31 lbs., and 4 lbs. with 42 lbs.

Ex. (2.) Let x = the men, y = the women, and z = the children.

1. Then, by first condition, $x+y+z=15.$
2. By second condition, $7x+3y+2z=65.$
3. Multiplying (1) by 2, $2x+2y+2z=30.$
4. Subtracting (3) from (2), $5x+y=35.$
5. Transposing, $5x=35-y.$
6. Dividing, $x=7-\frac{y}{5}.$
7. Substituting, $n=\frac{y}{5}.$
8. Clearing of fractions, $5n=y.$
9. Transposing, $y=5n.$
10. Substituting this last value of y for its value in (6), we have $x=7-\frac{5n}{5}=7-n.$
11. If we assume the value of n = 1, then the value of x = 6 = the men. And putting 6 for the value of x in
12. (4), we have $30+y=35.$
13. Transposing, $y=5$, the women.
14. Collecting terms, $15-(6+5)=4$, the children.

We therefore find the number of men to be 6, the women 5, the children 4.

If we were to assume the value of n = 2, then there would be 5 men, and 10 women. There could be no children, for $15-(5+10)=0$ children.

This question, therefore, will admit only of one answer.

Ex. (3.) Let x = the number of acres in the first farm, and y = the acres in the second.

1. Then, by conditions, $21x+17y=2000.$
2. Transposing, $21x=2000-17y.$
3. Dividing, $x=95+\frac{5-17y}{21}.$
4. Transposing, $x=95-\frac{17y-5}{21}.$

5. Substituting, $n=\frac{17y-5}{21}$.

6. Clearing of fractions, $21n=17y-5$.

7. Transposing, $17y=21n+5$.

8. Dividing, $y=\frac{21n+5}{17}$.

9. We assume the value of $n=3$; for this is the least number with which we can obtain y without a fraction; and we find $y=4$, and obtain the corresponding value of x in (2) by substituting for y its value in (8).

Having found the least value of $y=4$, and the corresponding value of $x=92$, we may find the remaining answers by adding 21 continually to the least value of y, and by subtracting 17 from the greatest value of x; which being done, we obtain the six following results:

If $n=3$, then $y=4$, and $x=92$.
$n=20$, " $y=25$, " $x=75$.
$n=37$, " $y=46$, " $x=58$.
$n=54$, " $y=67$, " $x=41$.
$n=71$, " $y=88$, " $x=24$.
$n=80$, " $y=109$, " $x=7$.

Therefore the first farm may contain 92, 75, 58, 41, 24, and 7 acres; and the second may contain 4, 25, 46, 67, 88, and 109 acres.

Ex. (4.) Let $x=$ the bushels of wheat, and $y=$ the bushels of barley.

1. Then, by conditions, $17x+11y=542$.

2. Transposing, $11y=542-17x$.

3. Dividing, $y=49-x-\frac{6x-3}{11}$.

4. Substituting, $n=\frac{6x-3}{11}$.

5. Clearing of fractions, $11n=6x-3$.

6. Transposing, $6x=11n+3$.

7. Dividing, $x=\frac{11n+3}{6}$.

If $n = 3$, we have $x = 6$, and $y = 40$.
$n = 9$, " " $x = 17$, " $y = 23$.
$n = 15$, " " $x = 28$, " $y = 6$.

That is, I can have 6 bushels of wheat, and 40 of barley; or, 17 bushels of wheat, and 23 of barley; or, 28 bushels of wheat, and 6 of barley.

Ex. (5.) Let $x =$ one part, and $y =$ the other.

1. Then, by conditions, $7x+11y=100$.
2. Transposing, $7x=100-11y$.
3. Dividing, $x=14-2y+\frac{2+3y}{7}$.
4. Substituting, $n=\frac{2+3y}{7}$.
5. Clearing of fractions, $7n=2+3y$.
6. Transposing, $3y=7n-2$.
7. Dividing, $y=\frac{7n-2}{3}$.

If $n = 2$, then $y = 4$, and $11\times4=44 =$ the least part; $100-44=56 =$ the greater part.

If we assume $n = 5$, and this is the next less number that will produce the value of y without a fraction, we shall find the value of $y = 11$; and this multiplied by $11=121 =$ to one of the parts into which 100 was to be divided, which is absurd.

Therefore 56 and 44 are the only two parts into which, according to the conditions of the question, 100 can be divided.

Ex. (6.) Let $x =$ the two-dollar bills, and $y =$ the three-dollar bills.

1. Then, by the conditions, $2x+3y=25$.
2. Transposing, $2x=25-3y$.
3. Dividing, $x=12-y-\frac{1-y}{2}$.
4. Changing terms, $x=12-2y+\frac{y+1}{2}$.
5. Substituting, $n=\frac{y+1}{2}$.

6. Clearing of fractions, $2n=y+1$.
7. Transposing, $y=2n-1$.
8. Putting the value of y into (4), $x=12-2(2n-1)+n$.
9. Uniting terms, $x=14-3n$.

Let $n=1$, then $x=11$, and $y=1$.
$n=2$, " $x=8$, " $y=3$.
$n=3$, " $x=5$, " $y=5$.
$n=4$, " $x=2$, " $y=7$.

That is, 11 two-dollar bills, and 1 three-dollar bill; or, 8 two-dollar bills, and 3 three-dollar bills; or, 5 two-dollar bills, and 5 three-dollar bills; or, 2 two-dollar bills, and 7 three-dollar bills.

PROOF:

$$11\times2+1\times3=25.$$
$$8\times2+3\times3=25.$$
$$5\times2+5\times3=25.$$
$$2\times2+7\times3=25.$$

Ex. (7.) Let x = the bushels of corn, and y = the bushels of wheat.

1. Then, by conditions, $70x+190y=920$.
2. Dividing by 10, $= 7x+19y=92$.
3. Transposing, $7x=92-19y$.
4. Dividing, $x=13-2y-\dfrac{1-5y}{7}$.
5. Changing terms, $x=13-2y-\dfrac{5y-1}{7}$.
6. Substituting, $n=\dfrac{5y-1}{7}$.
7. Clearing of fractions, $7n=5y-1$.
8. Transposing, $5y=7n+1$.
9. Dividing, $y=\dfrac{7n+1}{5}$.
10. Putting the value of y into (4), $x=13-\dfrac{2(7n+1)}{5}-n$.

Let $n=2$, then $x=13-6-2=5$, the corn.
$7\times5=35$; $92-35=57$; $57\div19=3$, wheat.

No other value of n will answer the conditions of the question. There were, therefore, 5 bushels of corn and 3 bushels of wheat.

Ex. (8.) Let $x =$ the number required.

Then $\frac{x-7}{17}$ and $\frac{x-13}{26} =$ whole numbers.

And, putting $\frac{x-7}{17}=p$, we have $x=17p+7$.

If this value of x be substituted for x in the second fraction, we shall have $\frac{17p+7-13}{26}=\frac{17p-6}{26}=$ whole numbers.

It is evident that $\frac{26p}{26}$ is a whole number.

Consequently $\frac{26p}{26}-\frac{17p-6}{26}=\frac{9p+6}{26}$, a whole number.

Or, $\frac{9p+6}{26}\times 3=\frac{27p-18}{26}=p+\frac{p+18}{26}$, a whole number.

By omitting p, we have $\frac{p+18}{26}$.

Let this number $\frac{p+18}{26}=n$.

$$p+18=26n.$$

$$p=26n-18.$$

If $n = 1$, then $p=(26\times 1)-18=8$.

Consequently $x=17p+7=(17\times 8)+7=143$, the number required. And this is the least number; for, if we assume $n = 2$, we shall have $x = 585$.

We therefore say that 143 is the least number that will answer the conditions of the question.

Ex. (9.) Let x, y, $z =$ the sheep, pigs and rabbits, respectively.

1. Then, by first condition, $x+y+z=20$.
2. By second condition, $31x+11y+z=400$.
3. Subtracting (1) from (2), $30x+10y=380$.
4. Dividing by 10, $3x+y=38$.
5. Transposing, $y=38-3x$.

By examining this last equation, we perceive that the value of x must be less than 13; we therefore first assume its value to be 12.

If $x = 12$, we have $y = 2$, and $z = 6$.
$x = 11$, " " $y = 5$, " $z = 4$.
$x = 10$, " " $y = 8$, " $z = 2$.
$x = 9$, " " $y = 11$, " $z = 0$.
$x = 8$, " " $y = 14$, " $z = -2$.

We perceive, therefore, there can be but three correct answers to the question: 12 sheep, 2 pigs, 6 rabbits; or, 11 sheep, 5 pigs, 4 rabbits; or, 10 sheep, 8 pigs, 2 rabbits.

Ex. (10.) Let x and y = the two numbers.

1. Then, by conditions, $7x+13y=71$.
2. Transposing, $7x=71-13y$.
3. Dividing, $x=10-y-\frac{6y-1}{7}$.
4. Substituting, $n=\frac{6y-1}{7}$.
5. Clearing of fractions, $7n=6y-1$.
6. Transposing, $6y=7n+1$.
7. Dividing, $y=\frac{7n+1}{6}$.

If we assume the value of n to be any number less than 5, we make y a fractional number. And, if we assume its value to be 5, we find the value of x to be a minus quantity.

Thus, $x=10-6-5=-1$.

And, as no number less than 5 will answer the conditions of the question, it is certain, from the above operation, that no number more than 5 will produce the numbers required.

Ex. (11.) In Art. 296 we have the following formula for obtaining two numbers, when the sum of their squares is given:

$$x=\left(\frac{2m}{m^2+1}\right)z, \text{ and } y=\left(\frac{m^2-1}{m^2+1}\right)z.$$

Let x and y = the two numbers required, $z = 35$, the square root of the given square, and m = any assumed number.

If $m = 1$, we have $x = 35$, and $y = 0$.
$m = 2$, " " $x = 28$, " $y = 21$.
$m = 3$, " " $x = 21$, " $y = 28$.
$m = 4$, " " $x = 16\frac{8}{17}$, " $y = 30\frac{15}{17}$.

We therefore find 28 and 21 the only integral numbers that will answer the conditions of the question.

Ex. (12.) In Art. 298, we have the following formula for obtaining two numbers, when we have the difference of the two squares given:

$$\text{Here, } x=\left(\frac{m^2+1}{2m}\right)z, \text{ and } y=\left(\frac{m^2-1}{2m}\right)z.$$

Let x and y = the two required numbers, $z = 39$, the square root of the difference of the squares of the number required, and m = any assumed number.

Then, if $m = 1$, we have $x = 39$, and $y = 0$.
$m = 3$, " " $x = 65$, " $y = 52$.

The numbers therefore are 65 and 52.

VARIATIONS, PERMUTATIONS, AND COMBINATIONS

Art. 299. (p. 267.)

Variations.

Ex. (2.) $5\times4\times3\times2=120$.

Ex. (3.) $8\times7\times6\times5=1680$,

Permutations.

Art. 300. (p. 268.)

Ex. (5.) $\dfrac{1\times2\times3\times4\times5\times6\times7}{1\times2}=2520$.

Ex. (6.) $\dfrac{1\times2\times3\times4\times5\times6\times7}{1\times2\times1\times2\times3\times4}=105$.

Ex. (7.)

$$\frac{1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11\times12\times13\times14\times15}{1\times2\times3\times4\times5\times1\times2\times1\times2\times3\times1\times2}=$$

454053600. *Ans.*

COMBINATIONS.

Art. 301. (p. 269.)

Ex. (9.) $\frac{8\times7\times6\times5\times4}{1\times2\times3\times4\times5}=56.$

Ex. (10.) $\frac{12\times11\times10\times9\times8\times7\times6\times5\times4\times3}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}=66.$

Ex. (11.) $\frac{40\times39\times38\times37\times36\times35}{1\times2\times3\times4\times5\times6}=3838380.$

Ex. (12.) $\frac{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}{1\times1\times2\times1\times2\times3\times1\times2\times3\times4}=12600.$

Ex. (13.)

$2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2=$ 65536; $65536-1=65535.$

LOGARITHMS.

Art. 311. (p. 285.)

Ex. (12.) What is the value of x in the following equation:

$$x=\frac{654\times320\times.3691}{87\times9\times.045}.$$

Log. 654 = 2.815578

" 320 = 2.505150

" .3691 = -1.567144

From 4.887872

Log. 87 = 1.939519
" 9 = 0.954243
" .045 =—2.653213
Take 1.546975
2192.28=3.340897

Ex. (13.) What is the value of *x* in the following equation:

$$x=\frac{.69\times7.5\times32.71\times.003}{87\times8968\times.0008}.$$

Log. .69 =—1.838849
" 7.5 = 0.875061
" 32.71 = 1.514681
" .003 =—3.477121
From —1.705712
" 87 = 1.939519
" 8968 = 3.952696
" .0008 =—4.903090
Take 2.795305
.000813=—2.910207

Ex. (14.) Multiply three hundred twenty-seven ten-thousandths by three hundred twenty-seven thousand.

Log. .0327 =—2.514548
" 327000 = 5.514548
10692.9= 4.029096

Ex. (15.) What is the product of one thousand and twenty-five, multiplied by three hundred twenty-seven ten-thousandths?

Log. 1025 = 3.010724
" .0327 =—2.514548
33.5175= 1.525272

Ex. (16.) Multiply .0716 by 1.326.

Log. .0716 =—2.854913
" 1.326 = 0.122544
.0949416=—2.977457

Ex. (17.) Multiply .0009 by .009.

Log. .0009	=—4.954243
" .009	=—3.954243
.0000081	=—6.908486

EVOLUTION BY LOGARITHMS.

Ex. (8.) $x=\frac{37}{223}\left(\frac{14.21\times.00208}{.035}\right)^{\frac{4}{3}}$.

Log. 14.21	= 1.152594
" .00208	=—3.318063
	—2.470657
" .035	=—2.544068
	—1.926589
	4
	3) —1.706356
	—1.902118
" 37	= 1.568202
	1.470320
" 223	= 2.348305
Ans. .132438	=—1.122015

Ex. (9.) $x=\frac{7}{11}\left(\frac{144}{237}\right)^{\frac{2}{3}}\cdot\left(\frac{703}{819}\right)^{\frac{3}{5}}$.

Log. 703	= 2.846955
" 819	= 2.913284
	—1.933671
	3
	5) —1.801013
	Add —1.960202
" 144	= 2.158362
" 237	= 2.374748
	—1.783614
	2

3) —1.567228

To —1.855742

—1.960202

—1.815944

“ 7 = 0.845098

0.661042

“ 11 = 1.041393

Ans. .416532=—1.619649

Ex. (10.) $x=\frac{345}{417}\left(\frac{872\times.0065}{.038\times4685}\right)^{\frac{3}{5}}$.

Log. 872 = 2.940516

“ .0065 =—3.812913

From 0.753429

“ .038 =—2.579784

“ 4685 = 3.670710

Take 2.250494

—2.502935

3

5) —5.508805

—1.101761

“ 345 = 2.537819

1.639580

“ 417 = 2.620136

Ans. .10457=—1.019444

Ex. (11.) $x=\frac{25}{476}\left(\frac{873}{956}\right)^3\cdot\left(\frac{278}{1973}\right)^{\frac{3}{4}}$.

Log. 278 = 2.444045

“ 1973 = 3.295127

—1.148918

3

4) —3.446754

Add —1.361688

Log. 873 = 2.941014

" 956 = 2.980458

—1.960556

3

To —1.881668

—1.243356

" 25 = 1.397940

0.641296

" 476 = 2.677607

Ans. .0091979=—3.963689

Ex. (12.) $x=\frac{17}{112}\left(\frac{13.73\times.0706}{.253}\right)^{\frac{3}{2}}$.

Log. .0706 =—2.848805

" 13.73 = 1.137671

—1.986476

" .253 =—1.403121

0.583355

3

2) 1.750065

1.875032

" 17 = 1.230449

3.105481

" 112 = 2.049218

Ans. 1.138= 1.056263

Ex. (13.) $x=\left(\frac{38.47\times.463}{.037\times576}\right)^{\frac{2}{3}}$.

Log. 38.47 = 1.585122

" .463 =—1.665581

From 1.250703

" .037 —2.568202

" 576 2.760422

Take 1.328624

$$\begin{array}{r} -1.922079 \\ 2 \\ \hline 3\,)\,-1.844158 \\ \hline \textit{Ans.}\ .887262=-1.948052\tfrac{2}{3}. \end{array}$$

Ex. (14.) $x=\left(\frac{475\times329\times1728}{128}\right)^{\frac{1}{3}}$.

$$\begin{array}{ll r} \text{Log.} & 475 & =2.676694 \\ \text{"} & 329 & =2.517196 \\ \text{"} & 1728 & =3.237544 \\ \cline{3-3} & & 8.431434 \\ \text{"} & 128 & =2.107210 \\ \cline{3-3} & & 3\,)\,6.324224 \\ \cline{3-3} & \textit{Ans.} & 128.2=2.108074 \end{array}$$

COMPOUND INTEREST.

Art. 312. (p. 292.)

Ex. (7.) $A=p(1+r)^t=16(1.05)^{30}$.

$$\begin{array}{ll r} \text{Log.} & 1.05 & =0.021189 \\ & & 30 \\ \cline{3-3} & & 0.635670 \\ \text{"} & 16 & =1.204120 \\ \cline{3-3} & \textit{Ans.} & \$69.15=1.839790 \end{array}$$

Ex. (8.) $A=p(1+r)^t=2000(1.08)^{11}$.

$$\begin{array}{ll r} \text{Log.} & 1.08 & =0.033424 \\ & & 11 \\ \cline{3-3} & & 0.367664 \\ \text{"} & 2000 & =3.301030 \\ \cline{3-3} & \textit{Ans.} & \$4663.31=3.668694 \end{array}$$

Ex. (9.) $A=p(1+r)^t=27.18(1.04)^8$.

Log. 1.04 $=0.017033$
8
0.136264
" 27.18 $=1.434249$
1.570513
3 months, 1.01 $=\ .004321$
Ans. \$37.56$=1.574834$

Ex. (10.) $A=p(1+r)^t=1728\left(1+\frac{.06}{4}\right)^{8\frac{1}{2}\times 4=34}$

Log. 1.015 $=0.006466$
34
0.219844
" 1728 $=3.237544$
Ans. \$2866.74—1728=\$1138.74$=3.457388$

Ex. (11.) $A=p(1+r)^t=18.29(1.04)^8$.

Log. 1.04 $=0.017033$
8
0.136264
" 18.29 $=1.262214$
1.398478
" 1.028=3 months, 12 days $=\ .011993$
Ans. \$25.73$=1.410471$

Ex. (12.) $p=\frac{A}{(1+r)^t}=\frac{800}{(1+.05)^7}$.

Log. 1.05 $=0.021189$
7
Arith. Complement of 0.148323
Is -1.851677
Log. 800 $=2.903090$
Ans. \$568.54$=2.754767$

Ex. (13.) $p=\frac{A}{(1+r)^t}=\frac{500}{\left(1+\frac{.06}{4}\right)^{36}}=\frac{500}{(1.015)^{36}}.$

$$\begin{array}{lr}
\text{Log. } 1.015 & =\ \ 0.006466 \\
 & 36 \\ \hline
\text{Arith. Com. of} & 0.232776 \\ \hline
\text{Is} & =-1.767224 \\
\text{Log. } 500 & =\ \ 2.698970 \\ \hline
\textit{Ans.}\ \$292.54= & 2.466194
\end{array}$$

Ex. (14.) $r=\left(\frac{A}{p}\right)^{\frac{1}{t}}=\left(\frac{1609.76}{800}\right)^{\frac{1}{12}}.$

$$\begin{array}{lr}
\text{Log. } 1609.76 & =3.206762 \\
\text{“ } 800 & =2.903090 \\ \hline
 & 0.303672\ (\tfrac{1}{12}. \\ \hline
 & 1.06=0.025306 \\
 & 1 \qquad\qquad\quad \\ \hline
 & .06.\quad \textit{Ans.}
\end{array}$$

Ex. (15.) $t=\frac{\text{Log.}\left(\frac{A}{p}\right)}{\text{Log. }(1+r)}=\frac{\left(\frac{\text{Log. }5007.43}{\text{Log. }3726}\right)}{(\text{Log. }1.03)}.$

$$\begin{array}{lr}
\text{Log. } 5007.43 & =3.699615 \\
\text{“ } 3726 & =3.571243 \\ \hline
 & 0.128372 \\
\text{“ } 1.03 & =\ .012837
\end{array}$$

$128372 \div 12837=10$ years. *Ans.*

Ex. (17.) $3p=p(1+r)^t=t=\frac{\text{Log. }3}{\text{Log. }(1+r)}=\frac{\text{Log. }3}{\text{Log.}(1+.05)}.$

$$\begin{array}{lr}
\text{Log. } 3 & =0.477121 \\
\text{“ } 1.05 & =0.021189 \\ \hline
\end{array}$$

$477121 \div 21189=22$ years, 188 days. *Ans.*

Ex. (18.) Log. 23,267498 $=7.366750$
" 17,068666 $=7.232199$
" 1,031465 $=0.134551(\frac{1}{10}$
1
.031465 per cent. *Ans.*

Ex. (19.) Log. 100,000000 $=8.000000$
" 23,267498 $=7.366750$
0.633250
" .031465 $=0.013455$

633250÷13455=47 years, 3 days.
1850+47=1897, May 3. *Ans.*

Ex. (20.) $A=p(1+r)^t=155(1+.035)^9$.
Log. 1.035 $=0.014940$
9
0.134460
" 155 $=2.190332$
\$211.24 $=2.324792$

\$211.24−155=\$56.24. *Ans.*

Ex. (21.) $A=p\left(1+\frac{r}{2}\right)^t=820\left(1+\frac{.045}{2}\right)^5=820(1+.0225)^5$.
Log. 1.0225 $=0.009663$
5
0.048315
" 820 $=2.913814$
\$916.49. *Ans.* $=2.962129$

Ex. (22.) $p=\frac{A}{\left(1+\frac{r}{2}\right)^t}=\frac{A}{(1+.0225)^5}=\frac{458.25}{1+.0225}$.
Log. 1.0225 $=0.009663$
5
0.048315
" 458.25 $=2.661102$
\$410.02. *Ans.* $=2.612787$

Ex. (23.) $r=\left(\frac{A}{p}\right)^{t}=\left(\frac{4663.34}{2000}\right)^{11}$.

Log. 4663.34 $=3.668696$

" 2000 $=3.301030$

$367666\,(\,11$

$1.08=\;.033424$

1

$.08$ per cent. *Ans.*

DISCOUNT AND PRESENT VALUE AT COMPOUND INTEREST.

ART. **313.** (p. 294.)

Ex. (1.) $p=\frac{S}{(1+r)^{t}}=\frac{600}{(1+.06)^{3}}$.

Log. 1.06 $=0.025306$

3

Take 0.075918

" 600 From $=2.778151$

\$503.77. *Ans.* $=2.702233$

Ex. (2.) $p=\frac{S}{(1+r)^{t}}=\frac{312.50}{(1+.045)^{2}}$.

Log. 1.045 $=0.019116$

2

Take 0.038232

" 312.50 From $=2.494850$

\$286.16. *Ans.* $=2.456618$

Ex. (3.) $p=\frac{S}{(1+r)^{t}}=\frac{1000}{(1+.05)^{4}}$.

Log. 1.05 $=0.021189$

4

0.084756

" 1000 3.000000

\$822.70. *Ans.* $=2.915244$

Ex. (4.) $d=S\left(1-\frac{1}{(1+r)^t}\right)=3700\left(1-\frac{1}{(1+.05)^{10}}\right)$.

Log. 1.05 =0.021189
10
0.211890
" 3700 =3.568202
\$2271.49, present worth, =3.356312
\$3700—\$2271.49=\$1428.51 discount, *Ans.*

Ex. (5.) $p=\frac{S}{(1+r)^t}=\frac{3456}{(1+.06)^5}$.

Log. 1.06 =0.025306
5
0.126530
" 3456 =3.538574
\$2582.52, *Ans.* =3.412044

Ex. (6.) $d=S\left(1-\frac{1}{(1+r)^t}\right)=3456\left(1-\frac{1}{(1+.06)^4}\right)$.

Log. 1.06 0.025306
4
0.101224
" 1000 3.000000
\$792.09. 2.898776
\$1000—\$792.09=\$207.91, *Ans.*

Ex. (7.) $\frac{100(1.02)^{20}-1}{.02}$.

Log. 1.02 0.008600
20
1.4859= 0.172000
1
.4859=—1.686547
" 100 =2.000000
1.686547
" .02 —2.301030
\$2429.49, amount, =3.385517

$$p=\frac{A}{1+r}=\frac{2429.49}{1+.08}$$

Log. 1.08 =0.033424

5

0.167120

" 2429.49 =3.385517

$1658.47, present worth, =3.218397

Ex. (8.) $A=p(1+r)^t=100(1+.02)^{12}$.

Log. 1.02 0.008600

12

0.103200

" 100 2.000000

$126.82, *Ans.* =2.103200

Ex. (9.) $p=\frac{A}{(1+r)^t}=\frac{400}{(1+.05)^3}$.

Log. 1.05 0.021189

3

0.063567

" 400 =2.602060

$345.53 2.538493

$500÷1.20=$416.66;

$416.66—$345.53=$71.13.

That is $500 is better by $71.13

Ex. (10.) $p=\frac{A}{(1+r)^t}=\frac{1000}{(1+.05)^6}$.

Log. 1.05 0.021189

6

0.127134

" 1000 3.000000

$746.21, *Ans.* =2.872866

DEPOSITS.

Art. 314. (p. 298.)

Ex. (4.) $A=\frac{a(a+r)((1+r)^t-1)}{r}=\frac{1000(1.03)((1.03)^{10}-1)}{.03}$.

Log. 1.03= .012837
10
1.3439=0.128370
1
“ .3439 =−1.536432
“ 1.03 = 0.012837
“ 1000 = 3.000000
“ .03=2.477121 Arith. Com. = 1.522879
11,807 inhabitants, *Ans.* = 4.072148

Ex. (5.) $A=\frac{a(a+r)((1+r)^t-1)}{r}=\frac{150(1.02)((1.02)^{50})-1}{.02}$.

Log. 1.02=.008600
50
2.6915=.430000
1
1.6915 =0.228272
1.02 =0.008600
150 =2.176091
.02=−2.301030 Arith. Com. =1.698970
$12,939, *Ans.* =4.111933

Ex. (6.) $t=\frac{\text{Log.}\left(\frac{Ar}{a(1+r)}\right)}{\text{Log.}\ (1+r)}+1=\frac{\text{Log.}\left(\frac{400\times.06}{47\times1.06}\right)}{\text{Log.}\ 1.06}+1.$

Log. 400×.06=24 From 1.380211
“ 1.06 0.025306
“ 47 1.672098
Take 1.697404

.48173 —1.682807
1.
1.48173= 0.170768
1.06 .025306

.170768÷.025306=6 years, 273 days.

Ex. (7.) $\frac{5000\times.045}{(1.045)(1.045)^6-1}$.

Log. 5000 3.698970
" .045 —2.653213
1.045=0.019116 2.352183
6
1.3022 0.114696
1
.3022 =—1.480294
1.045 = 0.019116
—1.499410
$712.48 for the son, = 2.852773

$\frac{5000\times.045}{1.045(1.045)^{11}-1}$.

Log. 5000 = 3.698970
.045 =—2.653213
From 2.352183
1.045=0.019116
11
1.62284=0.210276
1
.62284= —1.794367
1.045 = .019116
Take —1.813483
$345.71 for the daughter, = 2.538700

Ex. (8.) $\frac{300}{(1.04)((1.04)^{10}-1)}$.

Log. 300×.04=12 =1.079181

$$\begin{array}{llr}
1.04= & .017033 & \\
 & 10 & \\
\hline
1.4802 & =.170330 & \\
1 & & \\
\cline{1-1}
.4802 & & =-1.681422 \\
1.04 & & =0.017033 \\
\cline{3-3}
 & & 1.698455 \\
\cline{3-3}
 & \$24.02,8,\ \textit{Ans.} & 1.380726
\end{array}$$

Ex. (9.) $A=\frac{a(1+r)\,((1+r)^t-1)}{r}=\frac{10(1.015)\,((1.015)^{40}-1}{.015}$.

$$\begin{array}{llr}
\text{Log. } 1.015 = 0.006466 & & \\
\phantom{\text{Log. } 1.015 = 0.0064}40 & & \\
\hline
\text{“ } 1.814 = 0.258640 & & \\
\text{“ } 1. & & \\
\cline{1-1}
\text{“ } .814 & & =-1.910624 \\
\text{“ } 1.015 & & =0.006466 \\
\text{“ } 10 & & =1.000000 \\
\text{“ } .015 = -2.176091, \text{ Arith. C.} & & =1.823909 \\
\hline
 & \$550.81,\ \textit{Ans.} & =2.740999
\end{array}$$

Ex. (10.) $\frac{40(1.06)\,((1.06)^{10}-1)}{.06}$.

$$\begin{array}{llr}
\text{Log. } 1.06 = .025306 & & \\
\phantom{\text{Log. } 1.06 = .0253}10 & & \\
\hline
1.7908 = .253060 & & \\
1 & & \\
\cline{1-1}
.7908 & & =-1.898067 \\
1.06 & & 0.025306 \\
40 & & 1.602060 \\
.06 = -2.778151, \text{ Arith. Com.} & & 1.221849 \\
\hline
 & \$558.83 & 2.747282
\end{array}$$

$558.83—$550.81=$8.02 more, *Ans.*

EXPONENTIAL OR TRANSCENDENTAL EQUATIONS.

ART. 320. (p. 302.)

Ex. (9.) $$n=\frac{\text{Log.}\left(\frac{L}{a}\right)}{\text{Log. } r}=\frac{\text{Log.}\left(\frac{124.94}{78.39}\right)}{\text{Log. } 1.06}$$

Log. 124.94 =2.096701
78.39 =1.894261
0.202540

Log. 1.06=0.025306
Log. 202540 =5.306511
" 25306 =4.403157
Ans. 8 years, =0.903354

We do not add 1 to the years, because the number of years is one less than the number of terms. The same observation will apply to the 10th question.

Ex. (10.) $$\frac{\text{Log.}\left(\frac{4663.31}{2000}\right)}{\text{Log. } 1.08}.$$

Log. 4663.31 =3.668694
2000 =3.301030
0.367664

Log. 1.08=0.033424
" 367664 5.565451
33424 4.524058
Ans. 11 years, =1.041393

And 11 added to 1840=1851.

Ex. (11.) $$\frac{\text{Log.}\left(\frac{1609.76}{800}\right)}{\text{Log. } 1.06}.$$

Log. 1609.76 =3.206732
" 800 =2.903090
3.303642

Log. 1.06=0.025306.

" 303642 =5.482362

25306 =4.403223

12 years, *Ans.* =0.079139

Ex. (12.) $r=\left(\frac{A}{p}\right)^{\frac{1}{t}}-1=\left(\frac{4663.31}{2000}\right)^{\frac{1}{11}}-1.$

Log. 4663.31 =3.668694

" 2000 =3.301030

.367664(11

1.08 =.033242$\frac{2}{11}$

1

.08 per cent. *Ans.*

Ex. (13.) $n=$

$$\frac{\text{Log. }[a+(r-1)S]-\text{Log. }a}{r}=\frac{\text{Log. }[3+(3-1)295.23]-\text{Log. }3}{\text{Log. }3}.$$

$295.23\times\overline{3-1}+3=59049.$

Log. 59049 =4.771212

" 3 =0.477121

4.294091

4294091 =6.632871

477121 =5.678628

9 days. =0.954243

ANNUITIES.

ART. 329. (p. 306.)

Ex. (4.) $\frac{a\left((1+r)^{t}-1\right)}{r(1+r)^{t}}=\frac{500\left((1.05)^{20}-1\right)}{.05(1+.05)^{20}}.$

Log. 1.05=.021189

20

$$\begin{array}{lrr}
2.6532=.423780 & & \\
\underline{1} & & \\
1.6532 & = & 0.218325 \\
500 & = & \underline{2.698970} \\
 & \text{From} & 2.917295 \\
\hline
(1.05)^{20} & = & 0.423780 \\
.05 & = & -2.698970 \\
\hline
 & \text{Take} & -1.122750 \\
\hline
\$6230.81, \textit{ Ans.} & = & 3.794545
\end{array}$$

Ex. (5.) We first find the amount at compound interest of $6230.81, the sum deposited, for 10 years.

$$A=p(1+2)^t=6230.81(1+.05)^{10}.$$

$$\begin{array}{lr}
\text{Log. } 1.05 & =.021189 \\
 & \underline{10} \\
 & .211890 \\
\text{Log. } 6230.81 & =\underline{3.794545} \\
\quad \$10149.27 & =4.006435
\end{array}$$

We next find the amount of the annuity for ten years.

$$A=\frac{p((1+r)^t-1)}{r}=\frac{500((1.05)^{10}-1)}{.05}.$$

$$\begin{array}{llr}
\text{Log. } 1.05= & 0.021189 & \\
 & \underline{10} & \\
1.6289= & 0.211890 & \\
\underline{1.} & & \\
\text{Log. } .6289 & & =-1.798582 \\
\text{Log. } 500 & & =\underline{2.698970} \\
 & & =2.497552 \\
\text{Log. } .05 & & =\underline{-2,698970} \\
 & \$6289 & =3.798582
\end{array}$$

Hence $10149.27—$6289=$3860.27, *Ans.*

Ex. (6.) The answer to this question is the amount of an annuity of $500 for ten years, as found above. $6289.

Ex. (7.) $a=\frac{Ar(1+r)^t}{(1+r)^t-1}=\frac{1728\times.04(1.04)^{10}}{(1.04)^{10}-1.}.$

$$\begin{array}{llr}
\text{Log. } 1.04= & 0.017033 & \\
 & 10 & \\
\cline{2-2}
1.4802 & =0.170330 & \\
1 & & \\
\cline{1-2}
.4802=-1.681422, & \text{Arith. Com.} & =0.318578 \\
1728\times.04=69.12 & & =1.839604 \\
(1.04)^{10} & & =0.170330 \\
\cline{3-3}
 & \$213.09 & =3.328512
\end{array}$$

Ex. (8.) $t=\frac{\text{Log.}\left(\frac{a}{a-Ar}\right)}{\text{Log.}\ (1+r)}=\frac{\text{Log.}\ \frac{500}{500-4000\times.015}}{\text{Log.}\ (1.015)}.$

$$\begin{array}{lr}
\text{Log. } 500 & =2.698970 \\
500-(4000\times.015)=440 & =2.643453 \\
\cline{2-2}
 & .055517 \\
\cline{2-2}
1.015 & .006466
\end{array}$$

0.55517÷.006466=8 half-years, &c., or 4 years, 106 days.

Ex. (9.) $A=\frac{a((1+r)^t-1)}{r(1+r)^t}=\frac{90((1.01)^{40}-1)}{.01(1.01)^{10}}.$

$$\begin{array}{llr}
\text{Log. } 1.01= & 0.004321 & \\
 & 40 & \\
\cline{2-2}
\text{“ } 1.4888= & 0.172840 & \\
\phantom{\text{“ }}1 & & \\
\cline{1-1}
\text{“ } .4888 & & =-1.689131 \\
\text{“ } 90 & & =1.954243 \\
\cline{3-3}
 & & =1.643374 \\
\cline{3-3}
\text{“ } (1.01)^{40} & & =0.172840 \\
\text{“ } .01 & & =-2.000000 \\
\cline{3-3}
 & \$2954.84, \textit{Ans.} & =3.470534
\end{array}$$

INVOLUTION OF BINOMIALS.

ART. 330. (p. 310.)

Ex. (9.) What is the third power of $2a-b+c^2$?

Let $x=2a-b$, and $y=+c$.

Then, $(x+y)^3=x^3+3x^2y+3xy^2+y^3$.

$x^3=(2a-b)^3=8a^3-12a^2b+6ab^2-b^3$.

$3x^2y=3(2a-b)^2c^2=12a^2c^2-12abc+3b^2c^2$.

$3xy^2=3(2-b)c^4=6ac^4-3bc^4$.

$y^3=(c^2)^3=c^6$.

Therefore, $(2a-b+c^2)^3=8a^3-12a^2b+6ab^2-b^3+12a^2c^2-12abc^2+3b^2c^2+6ac^4-3bc^4+c^6$.

Ex. (10.) What is the fifth power of $4a-5b$?

Let $x=4a$, and $y=5b$.

Then, $(x-y)^5=x^5-5x^4y+10x^3y^2-10x^2y^3-5xy^4-y^5$.

$x^5=(4a)^5=1024a^5$.

$-5x^4y=5(4a)^4(5b)=6400a^4b$.

$+10x^3y^2=10(4a)^3(5b)^2=16000a^3b^2$.

$-10x^2y^3=10(4a)^2(5b)^3=20000a^2b^3$.

$+5xy^4=5(4a)(5b)^4=12500ab^4$.

$-y^5=(5b)^5=3125b^5$.

Therefore, $(4a-5b)^5=1024a^5-6400a^4b+16000a^3b^2-20000a^2b^3+12500ab^4-3125b^5$.

Ex. (11.) What is the sixth power of $3a^2-2b^3$?

Let $x=3a^2$, and $y=2b^3$.

Then, $(x-y)^6=x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.

$x^6=(3a^2)^6=729a^{12}$.

$-6x^5y=6(3a^2)^5(2b^3)=-2916a^{10}b^3$.

$+15x^4y^2=15(3a^2)^4(2b^3)^2=+4860a^8b^6$.

$-20x^3y^3=20(3a^2)^3(2b^3)^3=-4320a^6b^9$.

$+15x^2y^4=15(3a^2)^2(2b^3)^4=+2160a^4b^{12}$.

$-6xy^5=6(3a^2)(2b^3)^5=-576a^2b^{15}$.

$+y^6=(2b^3)^6=+64b^{18}$.

Therefore, $(3a^2-2b^3)^6=729a^{12}-2916a^8b^6+4860a^8b^6-4320a^6b^9+2160a^4b^{12}-576a^2b^{15}+64b^{18}$.

Ex. (12.) What is the fourth power of $m+n-p$?

Let $x=m+n$, and $y=-p$.

Then, $(x-y)^4=x^4-4x^3y+6x^2y^2-4xy^3+y^4$.

$x^4=(m+n)^4=m^4+4m^3n+6m^2n^2+4mn^3+n^4$.

$-4x^3y=4(m+n)^3(-p)=-4m^3p-12m^2np-12mn^2p-4n^3p$.

$6x^2y^2=6(m+n)^2(-p^2)=6m^2p^2+12mnp^2+6n^2p^2$.

$-4xy^3=4(m+n)(-p^3)=-4mp^3-4np^3$.

$y^4=(-p)^4=p^4$.

Therefore, $(m+n-p)^4=m^4+4m^3n+6m^2n^2+4mn^3+n^4-4m^3p-12m^2np-12mn^2p-4n^3p+6m^2p^2+12mnp^2+6n^2p^2-4mp^3-4np^3+p^4$.

Ex. (13.) What is the eighth power of m^2+n^3?

Let $x=m^2$, and $y=n^3$.

Then, $(x+y)^8=x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8$.

Let m be substituted for x, and multiply each of the powers by 2; and let n be substituted for y, and multiply each of its powers by 3.

Then, $(m^2-n^3)^8=m^{16}+8m^{14}n^3+28m^{12}n^6+56m^{10}n^9+70m^8n^{12}+56m^6n^{15}+28m^4n^{18}+8m^2n^{21}+n^{24}$.

Ex. (14.) What is the seventh power of $1+x^2$?

Let $x=1$, and $y=x^2$.

Then, $(x+y)^7=x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$.

For x and its powers in the above process substitute 1, and for y substitute x, and multiply each power of y by 2.

Then, $(1+x^2)^7=1+7x^2+21x^4+35x^6+35x^8+21x^{10}+7x^{12}+x^{14}$.

Ex. (15.) What is the second power of $a+b+c+d+e+f$?

Let $x=a+b+c$, and $y=d+e+f$.

Then, $(x+y)^2=x^2+2xy+y^2$.

And $x^2=(a+b+c)^2=a^2+2ab+2ac+b^2+2bc+c^2$.

$2xy=2(a+b+c)(d+e+f)=2ad+2ae+2af+2bd+2be+2bf+2cd+2cf$.

$y^2=(d+e+f)^2=d^2+2de+2df+e^2+2ef+f^2$.

Therefore, $(a+b+c+d+e+f)^2=a^2+2ab+2ac+b^2+2bc+c^2+2ad+2ae+2af+2bd+2be+2bf+2cd+2ce+2cf+d^2+2de+2df+e^2+2ef+f^2$.

Ex. (16.) What is the tenth power of a^3+b^3?

Let $x=a^3$, and $y=b^3$.

Then, $(x+y)^{10}=x^{10}+10x^9y+45x^8y^2+120x^7y^3+210x^6y^4+252x^5y^5+210x^4y^6+120x^3y^7+45x^2y^8+10xy^9+y^{10}$.

Change the x's and y's to a's and b's, and multiply each of their powers by 3, and we have

$$a^{30}+10a^{27}b^3+45a^{24}b^6+120a^{21}b^9+210a^{18}b^{12}+252a^{15}b^{15}+210a^{12}b^{18}+120a^9b^{21}+45a^6b^{24}+10a^3b^{27}+b^{30}.$$

Ex. (17.) What is the nth power of $a+b$?

$a^n+a^{n-1}b+a^{n-2}b^2+a^{n-3}b^3+a^{n-4}b^4 \dots . a^3b^{n-3}+a^2b^{n-2}+ab^{n-1}+b$.

We have omitted the coefficients.

Ex. (18.) What is the sixth power of $a-b+c$?

Let $x=a-b$, and $y=c$.

Then, $(x+y)^6=x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$.

And $x^6=(a-b)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.

$6x^5y=6(a-b)^5c=6(a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5)c=6a^5c-30a^4bc+60a^3b^2c-60a^2b^3c+30ab^4c-6b^5c$.

$15x^4b^2=15(a-b)^4(c^2)=15(a^4-4a^3b+6a^2b^2-4ab^3+b^4)c^2=15a^4c^2-60a^3bc^2+90a^2b^2c^2-60ab^3c^2+15b^4$.

$20x^3y^3=20(a-b)^3(c^3)=20(a^3-3a^2b+3ab^2-b^3)c^3=20a^3c^3-60a^2bc^3+60ab^2c^3-20b^3c^3$.

$15x^2y^4=15(a-b)^2(c^4)=15(a^2-2ab+b^2)c^4=15a^2c^4-30abc^4+15b^2c^4$.

$6xy^5=6(a-b)\,(c^5)=6ac^5-6bc^5$.

$y^6=c^6$.

Therefore, $(a-b+c)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-$

$6ab^5+b^6+6a^5c-30a^4bc+60a^3b^2c-60a^2b^3c+30ab^4c-6b^5c+15a^4c^2$
$-60a^3bc^2+90a^2b^2c^2-60ab^3c^2+15b^4+20a^3c^3-60a^2bc^3+60ab^2c^3+$
$20b^3c^3+15a^2c^4-30abc^4+15b^2c^4+6ac^5-6bc^5+c^6$.

Ex. (19.) What is the fourth power of a^5-x ?

Let $x=a^5$, and $y=x$.
Then, $(x-y)^4=x^4-4x^3y+6x^2y^2-4xy^3+y^4$.
Change x for a^5, and y for x. Multiply the powers of x by 5, and apply them to a. But x, which is substituted for y, may have the same powers as y.
Then, $(a^5-x)=a^{20}-4a^{15}x+6a^{10}x^2-4a^5x^3+x^4$.

Ex. (20.) What is the third power of $2a^2-3b^3$?

Let $x=2a^2$, and $y=-3b^3$.
Then, $(x-y)^3=x^3-3x^2y+3xy^2-y^3$.
And $x^3=(2a^2)^3=8a^6$.
" $-3x^2y=3(2a^2)^2(3b^3)=36a^4b^3$.
" $+3xy^2=3(2a^2)(3b^3)^2=54a^2b^6$.
" $-y^3=(3b^3)^3=27b^9$.
Hence, $(2a^2-3b^3)^3=8a^6-36a^4b^3+54a^2b^6=27b^9$.

BINOMIAL THEOREM.

ART. 332. (p. 314.)

Ex. (5.) What is the cube root of 7 ?

Here $\sqrt[3]{7}=\sqrt[3]{(8-1)}$.

Then, $P=8$; $Q=-\frac{1}{8}$; $m=1$; and $n=3$.

Hence, $P^{\frac{m}{n}}=8^{\frac{m}{n}}=8^{\frac{1}{3}}=+2=A$.

$$\frac{m}{n}AQ=\frac{1}{3}\times2\times-\frac{1}{8}=-\frac{2}{3.2^3}=-\frac{1}{3.2^2}=B.$$

$$\frac{m-n}{2n}BQ=-\frac{1}{3}\times-\frac{1}{3.2^2}\times-\frac{1}{8}=-\frac{1}{3.6.2^4}=C.$$

$$\frac{m-2n}{3n}CQ=-\frac{5}{9}\times-\frac{1}{3.6.2^4}\times-\frac{1}{8}=-\frac{5}{3.6.9.2^7}=D.$$

$$\frac{m-3n}{4n}DQ=-\frac{8}{12}\times-\frac{5}{3.6.9.2^7}\times-\frac{1}{8}=-\frac{5.8}{3.6.9.12.2^{10}}=E.$$

Therefore, $\sqrt[3]{7}=2-\frac{1}{3.2^2}-\frac{1}{3.6.2^4}-\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}-$, &c.

Ex. (6.) Expand $(1-a)^{\frac{2}{5}}$ into an infinite series.

Here $P=1$; $Q=-\frac{a}{1}$; $m=2$; and $n=5$.

Hence, $P^{\frac{m}{n}}=1^{\frac{m}{n}}=1^{\frac{2}{5}}=+1=A.$

$$\frac{m}{n}AQ=\frac{2}{5}\times-\frac{a}{1}=-\frac{2a}{5}=B.$$

$$\frac{m-n}{2n}BQ=-\frac{3}{10}-\frac{2a}{5}\times-\frac{a}{1}=-\frac{2.3a^2}{5.10}=C.$$

$$\frac{m-2n}{3n}CQ=-\frac{8}{15}\times-\frac{2.3a^2}{5.10}\times-\frac{a}{1}=-\frac{2.3.8a^3}{5.10.15}=D.$$

$$\frac{m-3n}{4n}DQ=-\frac{13}{20}\times-\frac{2.3.8a^3}{5.10.15}\times-\frac{a}{1}=-\frac{2.3.8.13a^4}{5.10.15.20}=E.$$

Therefore, $(1-a)^{\frac{2}{5}}=1-\frac{2a}{5}-\frac{2.3a^2}{5.10}-\frac{2.3.8a^3}{5.10.15}-\frac{2.3.8.13a^4}{5.10.15.20}-$, &c.

Ex. (7.) It is required to convert $\frac{1}{(1+x)^{\frac{1}{5}}}$, or its equal, $(1+x)^{-\frac{1}{5}}$, into an infinite series.

Let $P=1$; $Q=\frac{x}{1}$; $m=-1$; $n=-5$.

Then, $P^{\frac{m}{n}}=1^{-\frac{1}{5}}=1=A.$

$$\frac{m}{n}Q=-\frac{1}{5}\times\frac{x}{1}=-\frac{x}{5}=B.$$

$$\frac{m-n}{2n}BQ=\frac{-1-5}{10}\times-\frac{x}{5}\times\frac{x}{1}=+\frac{6x^2}{5.10}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{-1-10}{15}\times\frac{6x^2}{5.10}\times\frac{x}{1}=-\frac{6.11x^3}{5.10.15}=D.$$

$$\frac{m-3m}{4n}DQ=\frac{-1-15}{20}\times-\frac{6.11x^3}{5.10.15}\times\frac{x}{1}=+\frac{6.11.16x^4}{5.10.15.20}=E.$$

Therefore, $\dfrac{1}{(1+x)^{\frac{1}{5}}}=1-\dfrac{x}{5}+\dfrac{6x^2}{5.10}-\dfrac{6.11x^3}{5.10.15}+\dfrac{6.11.16x^4}{5.10.15.20}$, &c.

Ex. (8.) It is required to convert $(a-b)^{\frac{1}{4}}$ into an infinite series.

Let $P=a$; $Q=-\dfrac{b}{a}$ $m=1$; and $n=4$.

$$P^{\frac{m}{n}}=a^{\frac{1}{4}}=A.$$

$$\frac{m}{n}Q=\frac{1}{4}\times-\frac{b}{a}=-\frac{b}{4a}=B.$$

$$\frac{m-n}{2n}BQ=\frac{1-4}{8}\times-\frac{b}{4a}\times-\frac{b}{a}=-\frac{3b^2}{4.8a^2}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{1-8}{12}\times-\frac{3b^2}{4.8a^2}\times-\frac{b}{a}=-\frac{3.7b^3}{4.8.12a^3}=D.$$

$$\frac{m-3n}{4n}DQ=\frac{1-12}{16}\times-\frac{3.7b^3}{4.8.12a^3}\times-\frac{b}{a}=-\frac{3.7.11b^4}{4.8.12.16a^4}=E.$$

Therefore,

$$(a-b)^{\frac{1}{4}}=a^{\frac{1}{4}}\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-\text{, \&c.}\right).$$

INDETERMINATE COEFFICIENTS.

Art. 333. (p. 315.)

Ex. (2.) Expand $\dfrac{a^2}{a^2+2ax-x^2}$ into a series.

Assume $A+Bx+Cx^2+Dx^3+Ex^4=\dfrac{a^2}{a^2+2ax-x^2}$.

We first clear the equation of fractions.

$$\begin{array}{l} A+Bx+Cx^2+Dx^3+Ex^4. \\ \underline{a^2+2ax-x^2.\qquad\qquad\qquad\quad} \end{array}$$

$$a^2=\left\{\begin{array}{l} a^2A+a^2Bx+a^2Cx^2+a^2Dx^3+a^2Ex^4. \\ \qquad+2aAx+2aBx^2+2aCx^3+2aDx^4. \\ \qquad\qquad\qquad-Ax^2-Bx^3-Cx^4. \end{array}\right.$$

$a^2A+(a^2B+2aA)x+(a^2C+2aB-A)x^2+(a^2D+2aC-B)x^3+(a^2E+2aD-C)x^4.$

We next transpose a^2, and make each term $=0$.

1. Here $a^2A=a^2$, or $A=\frac{a^2}{a^2}=1$.

2. $a^2B+2aA=0$, or $a^2B=2a$, and $B=\frac{2a}{a^2}=\frac{2}{a}$.

3. $a^2C+2aB-A=0$, or $a^2C=A-2aB=1+4$, and $C=\frac{5}{a^2}$.

4. $a^2D+2aC-B=0$, or $a^2D=B-2aC=$

$$\frac{-2}{a}-\frac{2.5a}{a^2}-\frac{12}{a}, \text{ and } D=\frac{-12}{a^3}.$$

5. $a^2E+2aD-C=0$, or $a^2E=C-2aD=$

$$\frac{+5}{a^2}+\frac{24}{a^2}=\frac{-29}{a^2}, \text{ and } E=\frac{+29}{a^4}.$$

Therefore, substituting these values of A, B, C, D, E, &c., in the assumed equation, we have

$$\frac{a^2}{a+2ax-x^2}=1-\frac{2x}{a}+\frac{5x^2}{a^2}-\frac{12x^3}{a^3}+\frac{29x^4}{a^4}, \text{ \&c.,}$$

where the numerical coefficients form a recurring series, in which each term is equal to twice the preceding, added to that before it, as $12=\overline{2\times5}+2$; $29=\overline{12\times2}+5$, &c.

Ex. (3.) Expand $\sqrt{(a^2-x^2)}$ into a series.

Assume $A+Bx^2+Cx^4+Dx^6+Ex^8$, &c., $=(a^2-x^2)^{\frac{1}{2}}$.

$$\begin{array}{r}A+Bx^2+Cx^4+Dx^6+Ex^8.\\ A+Bx^2+Cx^4+Dx^6+Ex^8.\\ \hline\end{array}$$

$$a^2-x^2=\left\{\begin{array}{r}A^2+ABx^2+ACx^4+ADx^6+AEx^8.\\ +ABx^2+B^2x^4+BCx^6+BDx^8.\\ +ACx^4+BCx^6+C^2x^8.\\ +ADx^6+BDx^8.\\ +AEx^8.\end{array}\right.$$

1. Here $A^2=a^2$, $A=a$.

2. $2AB=-1$, and $B=-\frac{1}{2a}$.

3. $2AC+B^2=0$; $2AC=B^2$; $C=\frac{1}{8a^3}$.

4. $2AD+2BC=0$; $2AD=-2BC$; $D=\frac{1}{16a^5}$.

5. $2AE+2BD+C^2=0$; $AE=-\frac{5}{64a^6}$, and $E=\frac{-5}{128a^7}$.

Therefore, substituting these values of A, B, C, D, E, &c., in the assumed equation, we have $(a^2-x^2)^{\frac{1}{2}}=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$, &c., where the factors of the coefficients are the odd numbers in the numerator, and the even ones in the denominator. The exponents increase continually by 2, that of x in the numerator being always greater by 1 than that of a in the denominator; and the signs after the first term are all $-$.

Ex. (4.) Expand $\frac{1+2x}{1-x-x^2}$ into a series.

Assume $A+Bx+Cx^2+Dx^3+Ex^4$, &c., $=\frac{1+2x}{1-x-x^2}$.

$$\begin{array}{r}
A+Bx+Cx^2+Dx^3+Ex^4 \\
1-x-x^2 \\
\hline
A+Bx+Cx^2+Dx^3+Ex^4 \\
-Ax-Bx^2-Cx^3-Dx^4 \\
-Ax^2+Bx^3-Cx^4 \\
\hline
\end{array}$$

$1+2x=A+(B-A)x+(C-B-A)x^2+(D-C-B)x^3+(E-D-C)x^4$.

1. Here $A=1$.
2. $B-1=2$, or $B=+3$.
3. $C-3-1=0$, or $C=+4$.
4. $D-4-3=0$, or $D=4+3=+7$.
5. $E-7-4=0$, or $E=7+4=+11$, &c.

Therefore, substituting these values of A, B, C, D, E, &c., in the assumed equation, we have $\frac{1-2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4$, &c., where each coefficient is the sum of the two preceding ones.

Ex. (5.) Expand $\sqrt{(1-a)}$ into a series.

Assume $(1-a)^{\frac{1}{2}}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5$, &c.

Assume $(1-a)^{\frac{1}{2}}=A-Ba-Ca^2+Da^3+Ea^4-Fa^5-$, &c.

$$\begin{array}{r}
A-Ba-Ca^2-Da^3-Ea^4-Fa^5 \\
A-Ba-Ca^2-Da^3-Ea^4-Fa^5 \\
\hline
A^2-ABa-ACa^2-ADa^3-AEa^4-AFa^5 \\
-ABa+B^2a^2+BCa^3+BDa^4+BEa^5 \\
-ACa^2+BCa^3+C^2a^4+CDa^5 \\
-DAa^3+BDa^4+CDa^5 \\
-AEa^4+BEa^5 \\
-AFa^5 \\
\hline
\end{array}$$

$1-a=A^2-(2ABa)+(-2AC+B^2)a^2+(-2AD+2BC)a^3+(-2AE+2BD+C^2)a^4+(-2AF+2BE+2CD)a^5$.

Then, bringing all the terms to one side of the equation, and equating the coefficients, we have

1. $A^2-1=0$; $A^2=1$; $A=1$.
2. $-2AB+1=0$; therefore, $B=-\frac{1}{2}$.
3. $-2AC+B=0$; therefore, $C=\frac{1}{2.4}$.
4. $-2AD+2BC=0\ .\cdot.\ 2AD=-\frac{1}{2.4}$, and $D=\frac{3}{2.4.6}$.
5. $-2AE+2BD+C^2=0\ .\cdot.\ 2AE=-\frac{3}{2.4.6}+\frac{1}{64}$, and $E=\frac{3.5}{2.4.6.8}$.
6. $-2AF+2BE+2CD=0\ .\cdot.\ 2AF=-\frac{3.5}{2.4.6.8}+\frac{1}{64}$, and $F=\frac{3.5.7}{2.4.6.8.10}$, &c. $.\cdot.$

Therefore, substituting these values of A, B, C, &c., in the assumed equation, we obtain

$$\sqrt{(1-a)}=1-\frac{a}{2}-\frac{a^2}{2.4}-\frac{3a^3}{2.4.6}-\frac{3.5a^4}{2.4.6.8}-\frac{3.5.7a^5}{2.4.6.8.10}-\text{, \&c.}$$

Ex. (6.) Expand $\frac{1-x}{1-2x-3x^2}$ into a series.

Assume $\frac{1-x}{1-2x-3x^2}=A+Bx+Cx^2+Dx^3+Ex^4$.

Clear the equation of fractions by multiplying both sides by $1-2x-3x^2$, and we have

$$1-x=A+(B-2A)x+(C-2B-3A)x^2+(D-2C-3B)x^3+(E-2D-3C)x^4.$$

We transpose $1-x$, and equate the coefficients of the above terms with zero.

1. $A-1=0$; $A=1$.
2. $B-2A+1=0$; $B=2A-1=2-1=1$.
3. $C-2B-3A=0$; $C=2B+3A=2+3=5$.
4. $D-2C-3B=0$; $D=2C+3B=10+3=13$.
5. $E-2D-3C=0$; $E=2D+3C=26+15=41$.

Therefore, substituting these values of A, B, C, D, in the assumed equation, we have

$$\frac{1-x}{1-2x-3x^2}=1+x+5x^2+13x^3+41x^4+121x^5+365x^6.$$

Twice the coefficient of any term of this series, added to three times the preceding one, will give the following term. Thus, $\overline{2\times41}+\overline{3\times13}=121x^5$=6th term; $\overline{2\times121}+\overline{3\times41}=365=$ 7th term.

Ex. (7.) What is the expansion of $(a-b)^{\frac{1}{4}}$.

Assume $(A-Bx-Cx^2-Dx^3)^4=(a-b)^{\frac{1}{4}}$.

Raise this assumed quantity to the fourth power, and bring all the terms to one side of the equation. Make each term $=0$. Then substitute the values of A, B, C, D, &c., in the assumed equation, and we find

$$(a-b)^{\frac{1}{4}}=a^{\frac{1}{4}}\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-, \&c.\right).$$

Or,

This question may be solved thus:

Let $(a-b)^{\frac{1}{4}}=a^{\frac{1}{4}}\left(1-\frac{b}{a}\right)^{\frac{1}{4}}$. Expand $\left(1-\frac{b}{a}\right)^{\frac{1}{4}}$ into a series, and let the coefficients be represented by A, B, C, D, &c., as in the preceding examples, and we have

$$\left(1-\frac{b}{a}\right)^{\frac{1}{4}}=1-A\frac{b}{a}+B\frac{b^2}{a^2}-C\frac{b^3}{a^3}+D\frac{b^4}{a^4}.$$

By the Binomial Theorem, $A=\frac{1}{4}$; $B=\frac{1}{4}\times\frac{\frac{3}{4}}{2}=-\frac{1}{4}\times\frac{3}{8}$; $C=-\frac{1}{4}\times\frac{3}{8}\times\frac{\frac{7}{4}}{3}=\frac{1}{4}\times\frac{3}{8}\times\frac{7}{12}$; $D=\frac{1}{4}\times\frac{3}{8}\times\frac{7}{12}\times\frac{\frac{11}{4}}{4}=-\frac{1}{4}\times\frac{3}{8}\times\frac{7}{12}\times\frac{11}{16}$.

Substituting for A, B, C, &c., in the expansion, their values, we have

$$(a-b)^{\frac{1}{4}}=a^{\frac{1}{4}}\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-, \&c.\right)$$

Ex. (8.) It is required to expand $(a+x)^{-2}=\frac{1}{(a+x)^2}$.

Assume $A+Bx+Cx^2+Dx^3=\frac{1}{(a+x)^2}=\frac{1}{a^2+2ax+x^2}$.

$$\begin{array}{l}
A+Bx+Cx^2+Dx^3\\
a^2+2ax+x^2\\
\hline
Aa^2+Ba^2x+Ca^2x^2+Da^2x^3\\
\qquad +2Aax+2Bax^2+2Cax^3\\
\qquad\qquad +Ax^2+Bx^3
\end{array}$$

$$1=Aa^2+(Ba^2+2Aa)x+(Ca^2+2Ba+A)x^2+(Da^2+2Ca+B)x^3.$$

$Aa^2-1=0$; $Aa^2=1$; $A=\frac{1}{a^2}$.

$Ba^2+2A=0$; $Ba^2=-2Aa=\frac{2}{a}$; $B=\frac{2}{a}\times\frac{1}{a^2}=-\frac{2}{a^3}$.

$Ca^2+2Ba+A=0$; $Ca^2=-2Ba-A=-\frac{4}{a^2}+\frac{1}{a^2}=-\frac{3}{a^2}$; $C=-\frac{3}{a^2}\times\frac{1}{a^2}=\frac{3}{a^4}$.

$Da^2+2Ca+B=0$; $Da=-2Ca-B=-\frac{6}{a^3}+\frac{2}{a^3}=\frac{4}{a^3}$, $D=\frac{4}{a^3}\times\frac{1}{a^2}=-\frac{4}{a^5}$.

Substituting these values of A, B, C, D, &c., we have

$$\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{4x^3}{a^5}, \text{ \&c.}, =(a+x)^{-2} \text{ or } \frac{1}{(a+x)^2}.$$

Ex. (9.) It is required to expand $\frac{1}{(a+2x)^3}=$

$$\frac{1}{a^3+6a^2x+12ax^2+8x^3}.$$

Assume $A+Bx+Cx^2+Dx^3=\frac{1}{a^3+6a^2x+12ax^2+8x^3}.$

We proceed to multiply both sides of the equation by the denominator of the fraction, and we have: $Aa^3+(Ba^3+6Aa^2)x+(Ca^3+6Ba^2+12Aa)x^2+(Da^3+6Ca^2+12Ba+8A)x^3.$

By equating the coefficients of the different powers of x with zero, and then reducing the terms, we have

$$Aa^3-1=0. \qquad A=\frac{1}{a^3}.$$

$$Ba^3+6Aa^2=0. \qquad B=-\frac{6a^2}{1}\times\frac{1}{a^3}\times\frac{1}{a^3}=-\frac{6}{a^4}.$$

$$Ca^3+6Ba^2+12Aa=0. \qquad C=\left(\frac{36}{a^2}-\frac{12}{a^2}\right)\frac{1}{a^3}=+\frac{24}{a^5}.$$

$$Da^3+6Ca^2+12Ba+8A=0. \quad D=\left(\frac{144}{a^3}+\frac{72}{a^3}-\frac{8}{a^3}\right)\frac{1}{a^3}=\frac{80}{a^6}.$$

By substituting these values of A, B, C, D, &c., we have

$$\frac{1}{a^3}-\frac{6x}{a^4}+\frac{24x^2}{a^5}-\frac{80x^3}{a^6}=\frac{1}{(a+2x)^3}=\frac{1}{a^3+6ax^2+12a^2x+8x^6}.$$

Ex. (10.) It is required to find the expansion of

$$\frac{2}{(c+x)^2}=\frac{2}{c^2+2cx+x^2}.$$

We assume $A+Bx+Cx^2+Dx^3=\frac{2}{c^2+2cx+x^2}.$

Having multiplied both sides of this equation by the denominator, we obtain

$$2=Ac^2+(Bc^2+2Ac)x+(Cc^2+2Bc+A)x^2+(Dc^2+2Bc+B)x^3$$

By equating the coefficients, we have

1. $Ac^2-2=0$; $\qquad A=+\frac{2}{c^2}.$

2. $Bc^2+2Ac=0$; $\quad B=\frac{4c}{c^2}\times\frac{1}{c^2}=\frac{4c}{c^4}=\frac{4}{c^3}$.

3. $Cc^2+2Bc+A=0$; $C=\left(\frac{8c}{c^3}-\frac{2c}{c^3}\right)\frac{1}{c^2}=\frac{6c}{c^5}=+\frac{6}{c^4}$.

4. $Dc^2+2Cc+B=0$; $D=\left(\frac{12c}{c^4}-\frac{4c}{c^4}\right)\frac{1}{c^2}=\frac{8c}{c^6}=\frac{8}{c^5}$.

By substituting these values of A, B, C, D, &c., we have

$$\frac{2}{c^2}-\frac{4x}{c^3}+\frac{6x^2}{c^4}-\frac{8x^3}{c^5}, \text{ \&c.}, =\frac{2}{(c+x)^2}=\frac{2}{c^2+2cx+x^2}.$$

Ex. (11.) It is required to find the expansion of $\frac{a^2}{(a+2b)^3}$.

By dividing this expression into factors, &c., we have

$$\frac{a^2}{(a+2b)^3}=a^2(a+2b)^{-3}=a^2\times a^{-3}\left(1+\frac{2b}{a}\right)^{-3}=\frac{1}{a}\times\left(1+\frac{2b}{a}\right)^{-3}.$$

Expanding into a series, we have

$$\left(1+\frac{2b}{a}\right)^{-3}1+A\frac{2b}{a}+B\frac{4b^2}{a^2}+C\frac{8b^3}{a^3}+D\frac{16b^4}{a^4}.$$

By the Binomial Theorem, we find $A=-3$: $B=-3\times\frac{-4}{2}=6$;

$C=6\times\frac{-5}{3}=-10$; $D=-10\times\frac{-6}{4}=15$.

Substituting for A, B, C, &c., their values, we have

$$\frac{a^2}{(a+2b)^3}=\frac{1}{a}\left(1-\frac{6b}{a}+\frac{24b^2}{a^2}-\frac{80b^3}{a^3}+, \text{ \&c.}\right).$$

Ex. (12.) What is the value of $\frac{1}{(b^2+x)^{\frac{1}{2}}}$ in a series?

Assume $A+Bx+Cx^2+Dx^3=\frac{1}{(b^2+x)^{\frac{1}{2}}}$.

Square both sides of the equation, and multiply all the terms by the denominator of the fraction, and bring all the terms to one side of the equation, and we have

$-1+A^2b^2+(2ABb^2+A^2b^2)x+(2A^2Cb^2+Bb^2)x^2+(ADb^2+2BCb^2)x^3$.

$-1+A^2b^2+(ABb^2+A^2)x+(2ACb^2+2AB+B^2b^2)x^2$.
$+(ADb^2+2BCb^2+2AC+B)x^3+(AD+2BC)x^4$.

Then, substituting these values of A, B, C, &c., we have

$$\frac{1}{(b^2+x)^{\frac{1}{2}}}=\frac{1}{b}-\frac{x}{2b^3}+\frac{3x^2}{2.4b^5}-\frac{3.5x^3}{2.4.6b^7}+\frac{3.5.7x^4}{2.4.6.8b^9}, \text{ \&c.}$$

SUMMATION AND INTERPOLATION OF SERIES.

Art. 337. (p. 318.)

Ex. (3.) Required the several order of differences of the series $1^3, 2^3, 4^3, 5^3$.

1, 8, 27, 64, 125.
7, 19, 37, 61.
12, 18, 24.
6, 6.

Ex. (4.) Find the order of differences in the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c.

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}.$
$\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}.$
$\frac{1}{8}, \frac{1}{16}, \frac{1}{32}.$
$\frac{1}{16}, \frac{1}{32}.$
$\frac{1}{32}.$

Art. 388. (p. 319.)

Ex. (6.) It is required to find the first of the sixth order of differences of the series 3, 6, 11, 17, 24, 36, 50, 72, &c.

By adopting the formula of the fifth question, we obtain the following series:

$$3-6.6+\left(\frac{6.5}{2}\right)11-\left(\frac{6.5.4}{2.3}\right)17+\left(\frac{6.5.4.3}{2.3.4}\right)24-\left(\frac{6.5.4.3.2}{2.3.4.5}\right)36+\left(\frac{6.5.4.3.2.1}{2.3.4.5.6}\right)50=3-36+165-340+360-216+50=$$

-14, first term.

ART. 339. (p. 319.)

Ex. (8.) Required the ninth term of the series 1, 5, 15, 35, 70, &c.

1, 5, 15, 35, 70.
4, 10, 20, 35.
6, 10, 15.
4, 5.
1.

We find the first differences are 4, 10, 20, 35; the second are 6, 10, 15; the third 4, 5, and the fourth 1, &c.

Therefore, $d'=4$, $d''=6$, $d'''=4$, $d''''=1$.

By adopting the formula in seventh question, we have the following series:

$1+8.4+\left(\frac{8.7}{2}\right)6+\left(\frac{8.7.6}{2.3}\right)4+\left(\frac{8.7.6.5}{2.3.4}\right)1=1+32+168+$ $224+70=495$, the ninth term.

Ex. (9.) It is required to find the tenth term of the series 1, 3, 6, 10, 15, 21, &c.

1, 3, 6, 10, 15, 21.
2, 3, 4, 5, 6.
1, 1, 1, 1.

Here $d'=2$, $d''=1$, and $n=2$; therefore,

$a+\left(\frac{n-1}{1}\right)d'+\frac{(n-1)(n-2)}{1.2}d''=1+9.2+\left(\frac{9.8}{2}\right)1=$ $1+18+36=55$, the tenth term of the series.

ART. 341. (p. 320.)

Ex. (12.) Required the sum of 12 terms of the series 1, 4, 10, 20, 35, &c.

1, 4, 10, 20, 35.
3, 6, 10, 15.
3, 4, 5.
1, 1.

Let $n=12$, $a=1$, $d'=3$, $d''=3$, and $d'''=1$, and we have the following formula:

$$na+n.\left(\frac{n-1}{2}\right)d'+n.\left(\frac{n-1}{2}\right).\left(\frac{n-2}{3}\right)d''+n.\left(\frac{n-1}{2}\right).\left(\frac{n-2}{3}\right).\left(\frac{n-3}{4}\right)d'''12\times1+\left(\frac{12\times11}{2}\right)3+\left(\frac{12.11.10}{2.3}\right)3+\left(\frac{12.11.10.9}{2.3.4}\right)1=1365,$$ sum of twelve terms.

Ex. (15.) What is the number of cannon-shot in a square pile, the bottom row consisting of 25 shot?

Let $n=25$; and we have in the thirteenth question the following formula for the solution:

$$\frac{n(n-1).(2n+1)}{6}=\frac{25(25-1).(50+1)}{6}=5525 \text{ shot.}$$

Ex. (16.) I have 10 house-lots, whose sides measure 5, 6, 7, 8, 9, &c., rods, respectively. What is their value, at 25 cents per square foot?

25, 36, 49, 64.
11, 13, 15.
2, 2.

Let $n=10$, $d'=11$, $d''=2$, and $a=25$.

Then $na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2.3}=$

$10.25+\left(\frac{10.9}{2}\right)11+\left(\frac{10.9.8}{2.3}\right)2=985$ square rods.

$985\times272\frac{1}{4}=268166\frac{1}{4}$ square feet.

$268166\frac{1}{4}\times.25=\$67,041.56\frac{1}{4}$, its value at 25 cents per square foot.

In some editions of the Algebra this question is incorrect.

Ex. (17.) There are 5 cubical blocks of marble, whose sides measure respectively 2, 3, 4, 5 and 6 feet. What is their value, at $2.75 per cubic foot?

8, 27, 64, 125, 216.
19, 27, 61, 91.
18, 24, 30.
6, 6.

$$na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\ \ 3}d''+\frac{(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}d'''=$$

$$5\times8+\left(\frac{5.4}{2}\right)19+\left(\frac{5.4.3}{2.3}\right)18+\left(\frac{5.4.3.2}{2.3.4}\right)6=440 \text{ cubic feet.}$$

$$440\times2.75=\$1210.$$

Ex. (18.) What is the number of shot in a square pyramidical pile, whose side at the base contains 100 shot.

$$\frac{n(n-1).(2n+1)}{6}=\frac{100(100-1).(200-1)}{6}=338350 \text{ shot.}$$

Ex. (19.) What is the sum of 20 terms of the series 1^3, 2^3, 3^3, 4^3, 5^3, 6^3, &c.?

Let n = the number of terms.

Then, $\frac{n^2(n-1)^2}{4}=\frac{20^2(20-1)^2}{4}=44100$, sum of the terms.

Ex. (20.) What is the sum of 20 terms of the series 1^4, 2^4, 3^4, 5^4, 6^4, &c.?

1, 16, 81, 256, 625, 1296, 2401, 4096.
15, 65, 175, 369, 671, 1105, 1695.
50, 110, 194, 302, 434, 590.
60, 84, 108, 132, 156.
24, 24, 24, 24.

$$20.1+\frac{(20.19)}{2}15+\frac{(20.19.18)}{2.3}50+\frac{(20.19.18.17)}{2.3.4}60+$$
$$\frac{(20.19.18.17.16)}{2.3.4.5}24=3720960.$$

Art. 342. (p. 322.)

To find a fraction that will express the value of a geometrical series to infinity, we use the formula in page 242 of the Algebra.

$$S=\frac{a}{1-r}.$$

We therefore take the first term of the series for the numerator, and the algebraical difference between the ratio and 1 for the denominator of the fraction.

Ex. (2.) $\frac{1}{1-5}$.

Ex. (3.) $\frac{1}{1+a}$.

Ex. (4.) $\frac{\frac{h}{a}}{1-\frac{b}{a}}=\frac{h}{a-b}$.

Ex. (5.) $\frac{\frac{1}{x}}{1-\frac{1}{x}}=\frac{1}{x-1}$.

Ex. (6.) $\frac{1}{1-2}$.

Ex. (7.) $\frac{\frac{1}{a}}{1+\frac{x^2}{a^2}}=\frac{a}{a^2+x^2}$.

Ex. (8.) $\frac{1}{1-1}$.

Ex. (9.) $\frac{x}{1-\frac{x}{a}}=\frac{ax}{a-x}$.

Ex. (10.) $\frac{1}{1+\frac{x}{a}}=\frac{a}{a+x}$.

The succeeding questions may be performed by the following process:

Assume $\frac{q}{n}-\frac{q}{n+p}=\frac{pq}{n(n+p)}$; also, $\frac{q}{n(n+p)}=\frac{1}{p}\left(\frac{q}{n}-\frac{q}{n+p}\right)$.

Here $\frac{q}{n(n+p)}$ represents a series which is equal to $\frac{1}{p}$ multiplied into the difference of two series, represented by $\frac{q}{p}$ and $\frac{q}{n+p}$.

Applying this formula to the series following, we can find their value.

Ex. (11.) Required the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$, &c., to infinity.

Here $q=1$, $p=1$, 1, 2, 3, ∞, in succession.

NOTE.—This character, ∞, represents infinity.

$$\text{Then, } \frac{q}{n(n+p)}=\frac{1}{p}\left(\frac{q}{n}-\frac{q}{n+p}\right)=\left\{\begin{matrix} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots\frac{1}{\infty} \\ -\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\cdots\frac{1}{\infty} \end{matrix}\right\}=1.$$

Ex. (12.) Required the sum of the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}$, &c., to infinity.

Here $y=1$, $p=3$, $n=1, 2, 3 \ldots \infty$; hence the expression for the series becomes,

$$\frac{1}{3}\left\{\begin{array}{l}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\cdots\frac{1}{\infty}\\ \qquad\qquad -\frac{1}{4}-\frac{1}{5}-\frac{1}{\infty}-\frac{1}{\infty+1}, \text{ \&c.,}\end{array}\right\}=\frac{1}{3}\left(\frac{11}{6}\right)=\frac{11}{18}. \text{ Ans.}$$

Ex. (13.) Required the sum of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+$, &c., to infinity.

Here $q=1$, $p=2$, $n=1, 2, 3, 4$, &c. $\ldots \infty$.

Applying the formula, we have

$$\frac{1}{2}\left\{\begin{array}{l}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\cdots\frac{1}{\infty}\\ \qquad -\frac{1}{3}-\frac{1}{4}-\frac{1}{5}\cdots\frac{1}{\frac{1}{\infty}}-\frac{1}{\infty+1}\end{array}\right\}=\frac{1}{2}\left(\frac{3}{2}\right)=\frac{3}{4}. \text{ Ans.}$$

Ex. (14.) Required the sum of the series $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+$, &c., to infinity.

Here $q=2, 3, 5$, &c., $p=2$, $n=3, 5, 7 \ldots \infty$.

The formula becomes $\frac{1}{2}\left\{\begin{array}{l}\frac{2}{3}-\frac{3}{5}+\frac{4}{7}-\frac{5}{9}\cdots\mp\frac{\infty+1}{2\infty+1}\\ \quad -\frac{2}{5}+\frac{3}{7}-\frac{4}{9}\mp\frac{\infty+1}{2\,\infty+1}\mp\frac{\infty+1}{2\,\infty+3}\end{array}\right\}=$

$$\frac{1}{2}\left(\frac{2}{3}-1+1-1+1-, \text{ \&c.}\right)=\frac{1}{2}\left(\frac{2}{3}-\frac{1}{2}\right)=\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}. \text{ Ans.}$$

NOTE.—The 1's in the above question are obtained by reducing the fraction $\frac{1}{1+1}=\frac{1}{2}$; thus, $1\div\overline{1+1}=1-1+1-1+1-$, &c.

CUBIC EQUATIONS, CONTAINING ONLY THE THIRD AND SECOND POWERS.

Art. 343. (p. 328.)

Ex. (5.) Given $x^3+x^2=-4$ to find the values of x.

1. Conditions, $x^3+x^2=-4.$
2. Transposing, $-x^3=x^2+4.$
3. Multiplying by 4, $-4x^3=4x^2+16.$
4. Adding x^4 and $4x^3$ to both sides, $$x^4-4x^3+4x^2=x^4+8x^2+16.$$
5. Evolving, $x^2-2x=x^2+4.$
6. Cancelling, &c., $2x=-4.$
7. Dividing, $x=-2.$

To find the other values of x.

8. By conditions, $x^3+x^2=-4.$
9. Transposing, $x^3+x^2+4=0.$
10. Dividing by $x+2$, $x^2-x+2=0.$
11. Transposing, $x^2-x=-2.$
12. Comp. the square, $x^2-x+\frac{1}{4}=-2+\frac{1}{4}=-\frac{7}{4}.$
13. Evolving, $x-\frac{1}{2}=\frac{\pm\sqrt{-7}}{2}.$
14. Transposing, $x=\frac{1\pm\sqrt{-7}}{2}.$

Ex. (6.) Given $7x^2=x^3+36$ to find the values of x.

1. Conditions, $7x^2=x^3+36.$
2. Transposing, $-x^3=-7x^2+36.$
3. Multiplying by 4, $-4x^3=-28x^2+144.$
4. Adding x^4 and $4x^2$, $x^4-4x^3+4x^2=x^4-24x^2+144.$
5. Evolving, $x^2-2x=\pm(x^2-12).$
6. The plus value, $-2x=-12.$
7. Dividing by -2, $x=6.$
8. The minus value, $x^2-2x=-x^2+12.$

9. Transposing, $2x^2-2x=12$
10. Dividing, $x^2-x=6.$
11. Completing the square, $x^2-x+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}.$
12. Evolving, $x-\frac{1}{2}=\pm\frac{5}{2}.$
13. Transposing, $x=\pm\frac{5}{2}+\frac{1}{2}=3, \text{ or } -2.$

Hence x has three values, 6, 3, and -2, and each of them will answer the conditions of the question.

Ex. (7.) Given $x^3-4x^2=-9$ to find the values of x.

1. Conditions, $x^3-4x^2=-9.$
2. Transposing, &c., $-x^3=-4x^2+9.$
3. Multiplying by 4, $-4x^3=-16x^2+36.$
4. Adding x^4 and $4x^2$, $x^4-4x^3+4x^2=x^4-12x^2+36$
5. Evolving, $x^2-2x=\pm(x^2-6).$
6. The plus value, $-2x=-6.$
7. Dividing by -2, $x=3.$
8. The minus value, $x^2-2x=-x^2+6.$
9. Transposing, $2x^2-2x=6.$
10. Dividing, $x^2-x=3.$
11. Completing the square, $x^2-x+\frac{1}{4}=3+\frac{1}{4}=\frac{13}{4}.$
12. Evolving, $x-\frac{1}{2}=\frac{\pm\sqrt{13}}{2}.$
13. Transposing, $x=\frac{1\pm\sqrt{13}}{2}$

Ex. (8.) Given $2x^3=99-5x^2$ to find the values of x.

1. Conditions, $2x^3=99-5x^2.$
2. Dividing, $x^3=-\frac{5x^2}{2}+\frac{99}{2}.$
3. Multiplying by 22, $22x^3=-55x^2+1089.$
4. Adding x^4 and $121x^2$,

$$x^4+22x^3+121x^2=x^4+66x^2+1089.$$

5. Evolving, $x^2+11x=\pm(x^2+33).$

6. The plus value of x, $11x=33$.
7. Dividing, $x=3$.
8. The minus value of x, $x^2+11x=-x^2-33$.
9. Transposing, &c., $2x^2+11x=-33$.
10. Dividing, $x^2+\frac{11}{2}=-\frac{33}{2}$.
11. Comp. the square, $x^2+\frac{11}{2}+\frac{121}{16}=-\frac{33}{2}+\frac{121}{16}=-\frac{143}{16}$.
12. Evolving, $x^2+\frac{11}{4}=\frac{\pm\sqrt{-143}}{4}$.
13. Transposing, $x=\frac{-11\pm\sqrt{-143}}{4}$.

Ex. (9.) Given $4x^3+10x^2=125$ to find the values of x.

1. Conditions, $4x^3+10x^2=125$.
2. Transposing, $4x^3=-10x^2+125$.
3. Multiplying by 5, $20x^3=-50x^2+625$.
4. Adding x^4 and $100x^2$,

$$x^4+20x^3+100x^2=x^4+50x^2+625.$$

5. Evolving, $x^2+10x=\pm(x^2+25)$.
6. The plus value, $10x=25$.
7. Dividing, $x=2\frac{1}{2}$.
8. The minus value, $x^2+10x=-x^2-25$.
9. Transposing, $2x^2+10x=-25$.
10. Dividing, $x^2+5x=-\frac{25}{2}$.
11. Completing the square, $x^2+5x+\frac{25}{4}=-\frac{25}{2}+\frac{25}{4}=-\frac{25}{4}$.
12. Evolving, $x+\frac{5}{2}=\pm\frac{5}{2}$.
13. Transposing, $x=\pm\frac{5}{2}-\frac{5}{2}=-5$.

The two values of x are $2\frac{1}{2}$ and -5.

Ex. (10.) Given $x^3=8x^2+363$ to find the values of x.

1. Conditions, $x^3=8x^2+363$.
2. Multiplying by 12, $12x^3=96x^2+4356$.

3. Adding x^4 and $36x^2$, $x^4+12x^3+36x^2=x^4+132x^2+4356$.
4. Evolving, $x^2+6x=\pm(x^2+66)$.
5. The plus value, $6x=66$.
6. Dividing, $x=11$.
7. The minus value, $x^2+6x=-x^2-66$.
8. Dividing, &c., $x^2+3x=-33$.
9. Completing the square, $x^2+3x+\frac{9}{4}=-33+\frac{9}{4}=-\frac{123}{4}$.
10. Evolving, $x+\frac{3}{2}=\frac{\pm\sqrt{-123}}{2}$.
11. Transposing, $x=\frac{-3\pm\sqrt{-123}}{2}$

Ex. (11.) Given $37x^2=7x^3+144$ to find the values of x.

1. Conditions, $37x^2=7x^3+144$.
2. Multiplying by 4, &c., $-28x^3=-148x^2+576$.
3. Adding x^4 and $196x^2$,
   $x^4-28x^3+196x^2=x^4+48x^2+576$.
4. Evolving, $x^2-14x=\pm(x^2+24)$.
5. The plus value, &c., $-14x=24$.
6. Dividing by -14, $x=-1\frac{5}{7}$.
7. The minus value, $x^2-14x=-x^2-24$.
8. Dividing, &c., $x^2-7x=-12$.
9. Completing the square, $x^2-7x+\frac{49}{4}=-12+\frac{49}{4}=\frac{1}{4}$.
10. Evolving, $x-\frac{7}{2}=\pm\frac{1}{2}$.
11. Transposing, $x=\pm\frac{1}{2}+\frac{7}{2}=4$, or 3.

Hence the three values of x are 4, 3, $-1\frac{5}{7}$.

CUBIC EQUATIONS CONTAINING ONLY THE THIRD AND FIRST POWERS.

ART. 344. (p. 332.)

Ex. (4.) Given $x^3-7x=6$ to find the values of x.

1. Conditions, $x^3-7x=6.$
2. Multiplying by x, $x^4-7x^2=6x.$
3. Adding $4x^2$, $x^4-3x\quad 4x^2+6x.$
4. Completing the square, $x^4-3x^2+\frac{9}{4}=4x^2+6x+\frac{9}{4}.$
5. Evolving, $x^2-\frac{3}{2}=\pm 2x+\frac{3}{2}.$
6. Transposing the plus value, $x^2-2x=3.$
7. Completing the square, $x^2-2x+1=3+1=4.$
8. Evolving, $x-1=\pm 2.$
9. Transposing, $x=\pm 2+1=3$, or $-1.$
10. The minus value, $x^2-\frac{3}{2}=-2x-\frac{3}{2}.$
11. Cancelling, $x^2=-2x.$
12. Dividing, $x=-2.$

Hence the values of x are 3, -2, -1.

Ex. (5.) Given $x^3=37x+84$ to find the values of x.

1. Conditions, $x^3=37x+84.$
2. Transp. and mult. by x, $x^4-37x^2=84x.$
3. Adding $49x^2$, $x^4+12x^2=49x^2+84x.$
4. Completing the sq., $x^4+12x^2+36=49x^2+84x+36.$
5. Evolving, $x^2+6=\pm(7x+6).$
6. The plus value, $x^2=7x.$
7. Dividing, $x=7.$
8. The minus value, $x^2+6=-7x-6.$
9. Transposing, $x^2+7x=-12.$
10. Completing the square, $x^2+7x+\frac{49}{4}=-12+\frac{49}{4}=\frac{1}{4}.$
11. Evolving, $x+\frac{7}{2}=\pm\frac{1}{2}.$
12. Transposing, $x=\pm\frac{1}{2}-\frac{7}{2}=-4$, or $-3.$

Hence the values of x are 7, -4, -3.

Ex. (6.) Given $2x^3+7x=474$ to find the values of x.

1. By conditions, $2x^3+7x=474.$
2. Multiplying by x, $2x^4+7x^2=474x.$
3. Dividing, $x^4+\frac{7x^2}{2}=237x.$
4. Adding $36x^2$, $x^4+\frac{7x^2}{2}+36x^2=36x^2+237x.$
5. Uniting terms, $x^4+\frac{79x^2}{2}=36x^2+237x.$
6. Comp. the sq. $x^4+\frac{79x^2}{2}+\frac{6241}{16}=36x^2+237x+\frac{6241}{16}.$
7. Evolving, $x^2+\frac{79}{4}=\pm6x+\frac{79}{4}.$
8. Cancelling, $x^2=6x.$
9. Dividing, $x=6.$
10. The minus value, $x^2+\frac{79}{4}=-6x-\frac{79}{4}.$
11. Transposing, $x^2+6x=-\frac{79}{2}.$
12. Completing the sq., $x+6x+9=-\frac{79}{2}+9=-\frac{61}{2}.$
13. Evolving, $x+3=\pm\sqrt{-\frac{61}{2}}.$
14. Transposing, $x=-3\pm\sqrt{-\frac{61}{2}}.$

Another process to find the last values of x:

15. Conditions, $2x^3+7x=474.$
16. Transposing, $2x^3+7x-474=0.$
17. Dividing by $x-6$, $2x^2+12x+79=0.$
18. Transposing, $2x^2+12x=-79.$
19. Dividing, $x^2+6x=-\frac{79}{2}.$
20. Completing the square, $x^2+6x+9=-\frac{79}{2}+9=-\frac{61}{2}.$
21. Evolving, $x+3=\pm\sqrt{-\frac{61}{2}}.$
22. Transposing, $x=-3\pm\sqrt{-\frac{61}{2}}.$

Ex. (7.) Given $9x^3=169x+280$ to find the values of x.

1. By conditions, $9x^3=169x+280.$
2. Multiplying by x, $9x^4=169x^2+280x.$
3. Dividing, $x^4-\frac{169x^2}{9}=\frac{280x}{9}.$
4. Adding $25x^2$, $4x^4-\frac{169x^2}{9}+25x^2=25x^2+\frac{280x}{9}.$
5. Uniting terms, $x^4+\frac{56x^2}{9}=25x^2+\frac{280x}{9}.$
6. Comp. the square, $x^4+\frac{56x^2}{9}+\frac{784}{81}=25x^2+\frac{280x}{9}+\frac{784}{81}.$
7. Evolving, $x^2+\frac{28}{9}=\pm\left(5x+\frac{28}{9}\right).$
8. Cancelling, $x^2=5x.$
9. Dividing, $x=5.$
10. The minus value, $x^2+\frac{28}{9}=-5x-\frac{28}{9}.$
11. Transposing, $x^2+5x=-\frac{56}{9}.$
12. Completing the sq., $x^2+5x+\frac{25}{4}=-\frac{56}{9}+\frac{25}{4}=\frac{1}{36}.$
13. Evolving, $x+\frac{5}{2}=\pm\frac{1}{6}.$
14. Transposing, $x=\pm\frac{1}{6}-\frac{5}{2}=-\frac{7}{3}$, and $-\frac{8}{3}.$

Hence the values of x are 5, $-\frac{7}{3}$, $-\frac{8}{3}$.

Or, the last values of x may be thus found:

15. By conditions, $9x^3-169x-280=0.$
16. Dividing by $x-5$, $9x^2+45x+56=0.$
17. Transposing, $9x^2+45x=-56.$
18. Dividing, $x^2+5x=-\frac{56}{9}.$
19. Completing the sq., $x^2+5x+\frac{25}{4}=-\frac{56}{9}+\frac{25}{4}=\frac{1}{36}.$

20. Evolving, $x+\frac{5}{2}=\pm\frac{1}{6}$.

21. Transposing, $x=\pm\frac{1}{6}-\frac{5}{2}=-\frac{7}{3}$, and $-\frac{8}{3}$.

Ex. (8.) Given $x^3-3x=322$ to find the values of x.

1. Conditions, $x^3-3x=322$.
2. Multiplying by x, $x^4-3x^2=322x$.
3. Adding $49x^2$, $x^4+46x^2=49x^2+322x$.
4. Completing the sq., $x^4+46x^2+529=49x^2+322x+549$.
5. Evolving, $x^2+23=\pm(7x+23)$.
6. Cancelling, $x^2=7x$.
7. Dividing, $x=7$.
8. The minus values, $x^2+23=-7x-23$.
9. Transposing, $x^2+7x=-46$.
10. Completing the sq., $x^2+7x+\frac{49}{4}=-46+\frac{49}{4}=-\frac{135}{4}$.
11. Evolving, $x+\frac{7}{2}=\frac{\pm\sqrt{-135}}{2}=\frac{\pm3\sqrt{-15}}{2}$.
12. Transposing, $x=\frac{-7\pm3\sqrt{-15}}{2}$

PROBLEMS IN CUBIC EQUATIONS. (p. 333.)

Ex. (3.) Let x = the number of books.

1. Then, by conditions, $6x^3-51x^2=900$.
2. Adding x^4 and $9x^2$, &c.,
$$x^4+6x^3+9x^2=x^4+60x^2+900.$$
3. Evolving, $x^2+3x=x^2+30$.
4. Cancelling, $3x=30$.
5. Dividing, $x=10$, price of the books.

Ex. (4.) Let x = the number of dollars.

1. Then, by conditions, $3x^3+5x^2=272=16\times17$
2. Transposing, $3x^3=-5x^2+272=16\times17$
3. Dividing, $x^3=\frac{-5x^2}{3}+\frac{272}{3}=16\times\frac{17}{3}$

4. Multiplying by $\frac{17}{3}$, $\frac{17x^3}{3}=-\frac{85x^2}{9}+\frac{4624}{9}$.

5. Multiplying by 4, $\frac{68x^3}{3}=-\frac{340x^2}{9}+\frac{18496}{9}$.

6. Adding x^4 and $\left(\frac{34x}{3}\right)^2$ to both sides,

$$x^4+\frac{68x^3}{3}+\frac{1156x^2}{9}=x^4+\frac{272x^2}{3}+\frac{18496}{9}.$$

7. Evolving, $x^2+\frac{34x}{3}=x^2+\frac{136}{3}$.

8. Cancelling, $\frac{34x}{3}=\frac{136}{3}$.

9. Multiplying, $34x=136$.

10. Dividing, $x=\$4$.

Ex. (5.) Let x = the number of miles the vessel has sailed.

1. Then, by conditions, $19x=x^3+30$.
2. Transposing, $x^3-19x=-30$.
3. Multiplying by x, $x^4-19x^2=-30x$.
4. Adding $25x^2$, $x^4+6x^2=25x^2-30x$.
5. Completing the square, $x^4+6x^2+9=25x^2-30x+9$.
6. Evolving, $x^2+3=\pm(5x-3)$.
7. Transposing, $x^2-5x=-6$.
8. Completing the square, $x^2-5x+\frac{25}{4}=-6+\frac{25}{4}=\frac{1}{4}$.
9. Evolving, $x-\frac{5}{2}=\pm\frac{1}{2}$.
10. Transposing, $x=\pm\frac{1}{2}+\frac{5}{2}=3$, and 2.
11. The minus value, $x^2+3=-5x+3$.
12. Transposing, $x^2=-5x$.
13. Dividing, $x=-5$.

Hence the boat has gained either 3 or 2 miles, or it has lost 5 miles.

MISCELLANEOUS QUESTIONS.

(p. 334.)

Ex. (9.)

$$\begin{array}{lllll}
\multicolumn{5}{l}{\frac{1}{x^n}-\frac{na}{x^{n+1}}+\frac{n^2a^2}{2x^{n+2}}.} \\
\multicolumn{5}{l}{\underline{x^n+nax^{n-1}+\frac{n^2a^2x^{n-2}}{2}.}} \\
1- & \frac{na}{x}+ & \frac{n^2a^2}{2x^2}. & & \\
 & +\frac{na}{x}- & \frac{n^2a^2}{x^2}+ & \frac{n^3a^3}{2x^3}. & \\
 & & \underline{\frac{n^2a^2}{2x^2}-} & \underline{\frac{n^3a^3}{2x^3}+} & \underline{\frac{n^4a^4}{4x^4}.} \\
1 & & & & +\frac{n^4a^4}{4x^4}.
\end{array}$$

Ex. (10.)

$$\begin{array}{llllllll}
\multicolumn{8}{l}{1-x)1-x^8(1+x+x^2+x^3+x^4+x^5+x^6+x^7.} \\
\underline{1-x} & & & & & & & \\
 & x-x^8 & & & & & & \\
 & \underline{x-x^2} & & & & & & \\
 & & x^2-x^8 & & & & & \\
 & & \underline{x^2-x^3} & & & & & \\
 & & & x^3-x^8 & & & & \\
 & & & \underline{x^3-x^4} & & & & \\
 & & & & x^4-x^8 & & & \\
 & & & & \underline{x^4-x^5} & & & \\
 & & & & & x^5-x^8 & & \\
 & & & & & \underline{x^5-x^6} & & \\
 & & & & & & x^6-x^8 & \\
 & & & & & & \underline{x^6-x^7} & \\
 & & & & & & & x^7-x^8 \\
 & & & & & & & \underline{x^7-x^8}
\end{array}$$

Ex. (11.) $3\sqrt[3]{x-a^2}\,.\,4\sqrt{x^2-a}=3\sqrt[6]{x^2-2a^2x+a^4}\,.\,4\sqrt[6]{x^6-3ax^4+3a^2x^2-a^3}=12\sqrt[6]{(x^8-3ax^6-2a^2x^7+3a^2x^4+6a^3x^5+a^4x^6-a^3x^2-6a^4x^3-3a^5x^4+2a^5x+3a^6x^2-a^7)}$.

Ex. (12.) Given $\frac{41-35x}{105}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{2x-2\frac{1}{5}}{6}$ to find the value of x.

1. Conditions, $\frac{41-35x}{105}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{2x-2\frac{1}{5}}{6}$.

2. Clearing of fractions,

$152x-70x^2-82-105+30x^2=30x^2-20x-10-70x^2+147x-77$.

3. Collecting terms, $25x=100$.

4. Dividing, $x=4$.

Ex. (13.) Given $\sqrt{x+9}=1+\sqrt{x}$ to find the value of x.

1. Conditions, $\sqrt{x+9}=1+\sqrt{x}$.

2. Involving, $x+9=1+2\sqrt{x}+x$.

3. Collecting terms, $2\sqrt{x}=8$.

4. Dividing, $\sqrt{x}=4$.

5. Involving, $x=16$.

Ex. (14.) Given $\frac{3-2x}{1-2x}-\frac{5-2x}{7-2x}=1-\frac{4x^2-2}{7-16x+4x^2}$ to find the value of x.

1. Conditions, $\frac{3-2x}{1-2x}-\frac{5-2x}{7-2x}=1-\frac{4x^2-2}{7-16x+4x^2}$.

2. Clearing of fractions,

$21-20x+4x^2-5+12x-4x^2=7-16x+4x^2-4x^2+2$.

3. Uniting terms, $8x=-7$.

4. Dividing, $x=-\frac{7}{8}$.

Ex. (15.) Given $(\sqrt{x}+28).(\sqrt{x}+6)=(\sqrt{x}+38).(\sqrt{x}+4)$ to find the value of x.

1. Conditions, $(\sqrt{x}+28).(\sqrt{x}+6=(\sqrt{x}+38).(\sqrt{x}+4)$.

2. Multiplying, &c. $x+34\sqrt{x}+168=x+42\sqrt{x}+152$.

3. Uniting terms, $8\sqrt{x}=16$.

4. Dividing, $\sqrt{x}=2$.

5. Involving, $x=4$.

Ex. (16.) Given $(x-1)\sqrt{2x-x^2}=\frac{1}{2}$ to find the values of x.

1. Conditions, $(x-1)\sqrt{2x-x^2}=\frac{1}{2}$.
2. Involving, $(x^2-2x+1)(2x-x^2)=\frac{1}{4}$.
3. Multiplying, $-x^4+4x^3-5x^2+2x=\frac{1}{4}$.
4. Changing signs, $x^4-4x^3+5x^2-2x+\frac{1}{4}=0$.
5. Evolving, $x^2-2x+\frac{1}{2}=0$.
6. Transposing, $x^2-2x=-\frac{1}{2}$.
7. Completing the square, $x^2-2x+1=1-\frac{1}{2}=\frac{1}{2}$.
8. Evolving, $x-1=\pm\sqrt{\frac{1}{2}}$.
9. Transposing, $x=1\pm\frac{1}{\sqrt{2}}=\frac{\sqrt{2}\pm1}{\sqrt{2}}$.

Ex. (17.) Given $x-2\sqrt{x+2}=1+\sqrt[4]{x^3-3x+2}$ to find the values of x.

1. Conditions, $x-2\sqrt{x+2}=1+\sqrt[4]{x^3-3x+2}$.
2. Transposing, &c., $(x-1)-2\sqrt{x+2}=\sqrt[4]{(x-1)^2(x+2)}$.
3. Involving, $(x-1)-2\sqrt{x+2}=\sqrt{x-1}\,.\,\sqrt[4]{(x+2)}$.
4. Transposing,

$$(x-1)-\sqrt[4]{x+2}\,.\,\sqrt{x-1}=2\sqrt{x+2}.$$

5. Completing the square,

$$(x-1)-\sqrt[4]{x+2}\,.\,\sqrt{x-1}+\tfrac{1}{4}\sqrt{x+2}=2\sqrt{x+2}+\tfrac{1}{4}\sqrt{x+2}=\tfrac{9}{4}\sqrt{x+2}.$$

6. Evolving, $\sqrt{x-1}-\frac{1}{2}\sqrt[4]{x+2}=\pm\frac{3}{2}\sqrt[4]{x+2}$.
7. Uniting terms, $\sqrt{x-1}=2\sqrt[4]{x+2}$.
8. Involving, $x-1=4\sqrt{x+2}$.
9. Involving, $x^2-2x+1=16x+32$.
10. Uniting terms, $x^2-18x=31$.
11. Completing the square,

$$x^2-18x+81=31+81=112=16\times7.$$

12. Evolving, $x-9=\pm4\sqrt{7}$.
13. Transposing, $x=9\pm4\sqrt{7}$.
14. Taking the negative value, $\sqrt{x-1}=-\sqrt[4]{x+2}$.
15. Involving, $x^2-2x+1=x+2$.
16. Transposing, $x^2-3x=1$.

17. Completing the square, $x^2-3x+\frac{9}{4}=1+\frac{9}{4}=\frac{13}{4}.$

18. Evolving, $x-\frac{3}{2}=\pm\frac{\sqrt{13}}{2}.$

19. Transposing, $x=\frac{3\pm\sqrt{13}}{2}=\frac{1}{2}(3\pm\sqrt{13}).$

Ex. (18.) Given $\sqrt[m]{a+x}=\sqrt[2m]{x^2-5ax+b^2}$ to find the value of x.

1. Conditions, $\sqrt[m]{a+x}=\sqrt[2m]{x^2-5ax+b^2}.$
2. Involving, $a+x=\sqrt{x^2-5ax+b^2}.$
3. Squaring both terms, $a^2+2ax+x^2=x^2-5ax+b^2.$
4. Transposing, &c., $7ax=b^2-a^2.$
5. Dividing, $x=\frac{b^2-a^2}{7a}.$

Ex. (19.) Given $b^2=a^2+bx$ to find the value of x.

1. Conditions, $b^2=a^2+bx.$
2. Transposing, $bx=b^2-a^2.$
3. Dividing, $x=\frac{b^2-a^2}{b}.$

Ex. (20.) Given $\frac{5}{6}(x-a)-\frac{1}{5}(2x-3b)=10a+11b$ to find the value of x.

1. Conditions, $\frac{5}{6}(x-a)-\frac{1}{5}(2x-3b)=10a+11b.$
2. Clearing of fractions, &c.,
$$25x-25a-12x+18b=300a+330b.$$
3. Uniting terms, $13x=325a+312b.$
4. Dividing, $x=25a+24b.$

Ex. (21.) Given $\frac{3ac}{a+b}+\frac{ax}{(a+b)^2}+\frac{(2a+b)bx}{a(a+b)^2}=\frac{3cx}{b}+\frac{x}{a}$ to find the value of x.

1. Conditions, $\frac{3ac}{a+b}+\frac{ax}{(a+b)^2}+\frac{(2a+b)bx}{a(a+b)^2}=\frac{3cx}{b}+\frac{x}{a}.$
2. Clearing of fractions,
$$3a^2bc(a+b)+a^2bx+2ab^2x+b^3x=3acx(a+b)^2+a^2bx+2ab^2x+b^3x$$

3. Uniting terms, $ab=x(a+b)$.
4. Transposing, &c., $x(a+b)=ab$.
5. Dividing, $x=\frac{ab}{a+b}$.

Ex. (22.) Given $\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}=b^{\frac{1}{2}}$ to find the value of x.

1. Conditions, $\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}=b^{\frac{1}{2}}$.
2. Clearing of fractions,
$$\sqrt{a+x}+\sqrt{a-x}=\sqrt{b(a+x)}-\sqrt{b(a-x)}.$$
3. Squaring, &c., $2a+2\sqrt{a^2-x^2}=2ab-2b\sqrt{a^2-x^2}$.
4. Uniting terms, $(b+1)\sqrt{a^2-x^2}=ab-a$.
5. Involving, $(b^2+2b+1)(a^2-x^2)=a^2b^2-2a^2b+a^2$.
6. Multiplying,
$$a^2b^2-b^2x^2+2a^2b-2bx^2+a^2-x^2=a^2b^2-2a^2b+a^2.$$
7. Uniting terms, $x^2+2bx^2+b^2x^2=4a^2b$.
8. Evolving, $x+bx=2a\sqrt{b}$.
9. Dividing, $x=\frac{2a\sqrt{b}}{1+b}$.

Ex. (23.) Given $\frac{\sqrt{a+\sqrt{x}}}{\sqrt[4]{x}}+\frac{\sqrt{a-\sqrt{x}}}{\sqrt[4]{x}}=\sqrt[4]{x}$ to find the value of x.

1. Conditions, $\frac{\sqrt{a+\sqrt{x}}}{\sqrt[4]{x}}+\frac{\sqrt{a-\sqrt{x}}}{\sqrt[4]{x}}=\sqrt[4]{x}$.
2. Clearing of fractions,
$$\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}=\sqrt{x}.$$
3. Evolving, $2a+2\sqrt{a^2-x}=x$.
4. Transposing, $\sqrt{a^2-x}=\frac{x-2a}{2}$.
5. Clearing of fractions, &c., $4a^2-4x=x^2-4ax+4a^2$.
6. Cancelling, &c., $x^2=4ax-4x$.
7. Dividing, $x=4a-4$.
8. Separating into factors, $x=4(a-1)$.

Ex. (24.) Given $\sqrt{\frac{a^2}{x}+b}-\sqrt{\frac{a^2}{x}-b}=\sqrt{c}$ to find the value of x.

1. Conditions, $\sqrt{\frac{a^2}{x}+b}-\sqrt{\frac{a^2}{x}-b}=\sqrt{c}.$
2. Transposing, $\sqrt{\frac{a^2}{x}+b}=\sqrt{c}+\sqrt{\frac{a^2}{x}-b}.$
3. Involving, $\frac{a^2}{x}+b=c+\frac{a^2}{x}-b+2\sqrt{\frac{a^2c}{x}-bc}.$
4. Cancelling, &c., $2b-c=\sqrt{\frac{4a^2c}{x}-4bc}.$
5. Involving, $4b^2-4bc+c^2=\frac{4a^2c}{x}-4bc.$
6. Cancelling, &c., $4b^2x+c^2x=4a^2c.$
7. Transposing and dividing, $x=\frac{4a^2c}{4b^2+c^2}.$

Ex. (25.) Let $a=4$; $L=128$; $d=2$; $S=252$, and $n=$ the number of days required, and we have the following formula for its operation:

$$n=\frac{\text{Log. } L-\text{Log. } a}{\text{Log. } d}+1=\frac{\text{Log. } 128-\text{Log. } 4}{\text{Log. } 2}+1=$$

$$n=\frac{2.107210-.602060}{.301030}+1=6 \text{ days.}$$

In performing this question, it is not necessary to make use of the sum of the series.

Ex. (26.) Let $a=1000$; $L=5062.50$; $d=1.5$, and $n=$ the number of terms.

$$n=\frac{\text{Log. } 5062.50-\text{Log. } 1000}{\text{Log. } 1.5}+1=\frac{3.704365-3.000000}{.176091}+1$$

$=5$ sons.

Ex. (27.) Let $a=3$, $L=\frac{1}{9}$, $S=4\frac{4}{9}$, and $n=$ the number of terms.

$$\text{Then, } n=\frac{\text{Log. } a-\text{Log. } L}{\text{Log. }(S-L)-\text{Log. }(S-a)}+1 \;=$$

$$n=\frac{\text{Log. } 3-\text{Log. } \frac{1}{9}}{\text{Log. }((4\tfrac{4}{9}-\tfrac{1}{9})=\tfrac{13}{3})-(\text{Log.}(4\tfrac{4}{9}-3)=\tfrac{13}{9})}+1.$$

$\text{Log. } 3 = .477121$

$\text{“ } \frac{1}{9} = -1.045757$

1.431364

$\text{“ } \frac{13}{3} = .636822$

$\text{“ } \frac{13}{9} = .159700$

$.477122$

$1.431364 \div .477122=3$; $3+1=4$ = number of terms.

Ex. (28.) Let $a=\frac{1}{5}$, $r=7$, $L=3361\frac{2}{5}$, and $n=$ the number of terms.

$$\text{Then } n=\frac{\text{Log. } L-\text{Log. } a}{\text{Log. } r}+1=$$

$$\frac{\text{Log. } 3361\frac{2}{5}-\text{Log. } \frac{1}{5}}{\text{Log. } 7}+1.$$

$\text{Log. } 3361\frac{2}{5} = 3.526519$

$\text{“ } \frac{1}{5} = -1.301030$

4.225489

$\text{“ } 7 = .845098$

$4.225489 \div .845098=5$, $5+1=6$, number of terms.

Ex. (29.) By subtracting the first term from the last, we have the difference of the extremes; thus, $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$.

And, by dividing the difference of the extremes by the number of terms less 1, we have the common difference, $\frac{1}{6}-(5-1)=\frac{1}{24}=d$. $\frac{1}{3}+\frac{1}{24}=\frac{3}{8}$; $\frac{3}{8}+\frac{1}{24}=\frac{5}{12}$; $\frac{5}{12}+\frac{1}{24}=\frac{11}{24}$.

The means, therefore, are $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

Ex. (30.) $S=\frac{n}{2}(2a+dn-d)=$

$S=\frac{200}{2}(2+400-2)=100(400)=40{,}000$, the sum.

Ex. (31.) $S=\frac{n(a+b)}{2}=\frac{10(-7+12)}{2}=25$, sum of the series.

Ex. (32.) $S=\frac{n}{2}(2a+(n-1)-d)=4(40+(8-1)-5)=20$ m.

Ex. (33.) $d=\frac{2S-2an}{n(n-1)}=\frac{247-200}{20(20-1)}=\frac{47}{380}=$ common differ.

Ex. (34.) It is evident that the number of years must be infinite. For, if the first term of a descending series be 20, and the second 19, and every succeeding term $\frac{19}{20}$ of the former, it is certain that it will require an infinite number of terms to amount to 400. See Algebra, Art. 285.

Ex. (35.) Let $x=$ the wine, and $y=$ the water.

1. Then, by first condition, $x+6:y+6::7:6$.
2. Multiplying extremes, &c., $6x+36=7y+42$.
3. Transposing, $6x-7y=6$.
4. By second condition, $x-6:y-6::6:5$.
5. Multiplying extremes, &c., $5x-30=6y-36$.
6. Transposing, $5x-6y=-6$.
7. By subtracting 6 times (6) from 5 times (3), we have, $y=66$, water.
8. Putting y into (3), $6x-462=6$.
9. Transposing, $6x=468$.
10. Dividing, $x=78$, wine.

Ex. (36.) Let $x=$ the side of the larger stack, and $y=$ the side of the smaller.

1. First condition, $x^3y+y^3x=41\times20=820$.
2. Second condition, $x^2-y^2=9$.
3. The (1) divided by xy, $x^2+y^2=\frac{820}{xy}$.
4. The (3) involved, $x^4+2x^2y^2+y^4=\frac{(820)^2}{x^2y^2}$.
5. The (2) involved, $x^4-2x^2y^2+y^4=81$.
6. Subtracting (5) from (4), $4x^2y^2=\frac{(820)^2}{x^2y^2}-81$.
7. Clearing of fractions, $4x^4y^4=(820)^2-81x^2y^2$

8. Dividing by 4, $x^4y^4=(410)^2-\frac{81x^2y^2}{4}$.

9. Transposing, $x^4y^4+\frac{81x^2y^2}{4}=(410)^2$.

10. Comp. the square, $x^4y^4+\frac{81x^2y^2}{4}+\left(\frac{81}{8}\right)^2=(410)^2+\left(\frac{81}{8}\right)^2$.

11. Involving terms,

$$x^4y^4+\frac{81x^2y^2}{4}+\frac{6561}{64}=168100+\frac{6561}{64}=\frac{10764961}{64}.$$

12. Evolving, $x^2y^2+\frac{81}{8}=\pm\frac{3281}{8}$.

13. Transposing, $x^2y^2=\pm\frac{3281}{8}-\frac{81}{8}=\frac{3200}{8}=400$.

14. Evolving, $xy=20$.

15. Putting xy into (3), $x^2+y^2=\frac{820}{xy}=\frac{820}{20}=41$.

16. The (2), $x^2-y^2=9$.

17. Adding (15) and (16), $2x^2=50$.

18. Dividing, $x^2=25$.

19. Evolving, $x=5$.

20. Putting x into (14), $5y=20$.

21. Dividing, $y=4$.

$5^3\times4=500$; $500\div20=L.25$ = price of the greater.

$4^3\times5=320$; $320\div20=L.16$ = price of the smaller.

Ex. (37.) Let x = the sum paid daily.

1. Interest for one day, $\frac{x\times.06\times1}{365}=\frac{.06x}{365}$.

2. Interest for 364 days, $\frac{x\times.06\times364}{365}=\frac{21.84}{365}$.

3. Sum of the extremes, $\frac{.06x}{365}+\frac{21.84}{365}=\frac{21.9x}{365}$.

4. Half the extremes, $\frac{21.9x}{365}\div2=\frac{10.95}{365}$.

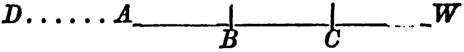
5. Multiplying, $\frac{10.95}{365}\times364=\frac{3985.8x}{365}$.

6. Add'g principal, $365x+\frac{3985.8x}{365}=1000\times1.06=1060$.

7. Clearing of fractions, $137210.8x = 386900$.

8. Dividing, $x = \$2.819748$.

Ex. (38.)



Let A be the place where the first traveller overtook the geese, and W the place where he met the wagon; also, let D be the place where the second traveller was when the first was at A, B the place where he overtook the geese, and C the place where he met the wagon.

Let y be the rate they travel per hour. The distance between AB is 5 miles, and the geese took $\frac{2.5}{3} = \frac{10}{3}$ hours in going it; in which time B had gone over the space $DB = \frac{10y}{3}$ miles; therefore DA, the constant distance between the travellers, is

$$\frac{10y}{3} - 5 = \frac{10y - 15}{3} \text{ miles.}$$

Again, $AW = 2y$, and, as A is 50 miles from Boston, W is $50 - 2y$ from it, and C is $31 + \frac{2y}{3}$ miles from it; whence CW is $= \frac{8y - 57}{3}$ miles; $\frac{4}{9}$ of this, or $\frac{32y - 228}{27}$, is the time which the wagon takes in going from W to C, in which time the first traveller will have gone $\frac{32y^2 - 288y}{27}$ miles. Now, this added to CW gives the distance between the travellers $= \frac{32y^2 - 156y - 513}{27}$ miles.

1. Consequently, $\frac{32y^2 - 156y - 513}{27} = \frac{90y - 135}{27}$.

2. Clearing of fractions and uniting terms, we have,

$$32y^2 - 246y = 378.$$

3. Dividing by 32, $y^2 - \frac{123y}{16} = \frac{378}{32}$.

4. Completing the square,

$$y^2 - \frac{123y}{16} + \frac{15129}{1024} = \frac{378}{32} + \frac{15129}{1024} = \frac{27225}{1024}.$$

5. Evolving, $y-\frac{123}{32}=\pm\frac{165}{32}$.

6. Transposing, $y=\pm\frac{165}{32}+\frac{123}{32}=9$ miles, or $-\frac{21}{16}$ miles.

Hence they travel at the rate of 9 miles per hour, and the distance between them is $\frac{10y}{3}-5=30-5=25$ miles from Boston.

Ex. (39.) Let x = the sum John deposits; then $1000-x$ will equal the sum Nathan deposits.

Then, by formula, page 289 of the Book,

$$A=p(1+r)^t.$$

1. Therefore, $x(1.05)^{11}=\overline{1000-x}(1.05)^6$.
2. Involving, $1.71032x=1340.09-1.34009x$.
3. Transposing, $3.05041x=1340.09$.
4. Dividing, $x=\$439.30$, John.
5. Subtracting, $1000-439.30=\$560.70$, Nathan.

Ex. (40.) Let x = the width of the ditch. The contents of the ditch will be equal to the circumference of the garden multiplied by 5 and x, $=100\times4\times5\times x=2000x$, and to this we must add 20 times the square of x for the corners of the ditch. The quantity of earth to be placed on the garden is equal to its surface multiplied by 2, $=100\times100\times2=20000$.

1. Therefore, $20x^2+2000x=20000$.
2. Dividing, $x^2+100x=1000$.
3. Comp. the sq., $x^2+100x+2500=1000+2500=3500$.
4. Evolving, $x+50=\pm59.1+$.
5. Transposing, $x=\pm59.1-50=9.1+$ feet.

Ex. (41.) Let x = the time B could reap the field.

1. Then, $\frac{3}{x}(1000)=\frac{3000}{x}$, = B received.
2. And $\frac{1}{2}(1000)=500$, = A received.
3. " $\frac{1}{2}-\frac{3}{x}=\frac{x-6}{2x}$, and $\frac{x-6}{2x}(1000)=\frac{500x-3000}{x}$, = C received.

4. $\frac{2070}{19}=$ B forfeited.

5. $\frac{500x-3000}{x}-\frac{2070}{19}=\frac{7430x-57000}{19x}$, $=$ A forfeited.

6. $\frac{3000}{x}:\frac{2070}{19}::300:\frac{7430x-57000}{19x}$.

7. $x^2-\frac{7430x}{207}=\frac{57000}{207}$.

8. Completing the square,

$$x^2-\frac{7430x}{207}+\frac{13801225}{42849}=-\frac{5700}{207}+\frac{13801225}{42849}=\frac{2002225}{42849}.$$

9. Evolving, $x-\frac{3715}{207}=\pm\frac{1415}{207}$.

10. Transposing,

$$x=\pm\frac{1415}{207}+\frac{3715}{207}=11\tfrac{1}{9}\text{ days, or }24\tfrac{18}{23}\text{ days, B's days.}$$

11. Dividing,

$$\frac{x-6}{2x}\div2=\frac{x-6}{4x}=\frac{23}{200}=8\tfrac{16}{23}\text{, or }5\tfrac{5}{18}\text{, C's days.}$$

Ex. (42.) Let $x=$ the number of miles per hour.

Then, $\frac{105}{x}=$ the number of hours.

1. Therefore, $\frac{105}{x-2}-6=\frac{105}{x}$.
2. Clearing of fractions, $105x-6x^2+12x=105x-210$.
3. Reducing terms, $x^2-2x=35$.
4. Completing the square, $x^2-2x+1=35+1=36$.
5. Evolving, $x-1=6$.
6. Transposing, $x=6+1=7$.

Ex. (43.) Let $x=9$, the difference between the hypothenuse and the base. The difference between the hypothenuse and perpendicular is 3 feet.

Let $x=$ the base, $x+6=$ the hypothenuse, and $x+3=$ the perpendicular.

1. Then, $(x+3)^2+x^2=(x+6)^2$.
2. Involving terms, $x^2+6x+9+x^2=x^2+12x+36$.

3. Cancelling, &c., $x^2-6x=25.$
4. Completing the sq. $x^2-6x+9=25+9=36.$
5. Evolving, $x-3=6.$
6. Transposing, $x=6+3=9$, the base.
7. Adding, $9+3=12$, the perpendicular.
8. " $9+6=15$, the hypothenuse.

Ex. (44.) Let x = the number of copper coins, and $24-x$ = the silver coins.

1. Then, $\overline{24-x}\times x=24x-x^2$ = value of the copper.
2. And $x\times\overline{24-x}=24x-x^2$ = value of the silver.
3. Conditions, $\overline{24x-x^2+24x-x^2}=48x-2x^2=216.$
4. Dividing, $24x-x^2=108.$
5. Transposing, $x^2-24x=-108.$
6. Completing the sq., $x^2-24x+144=-108+144=36.$
7. Evolving, $x-12=6.$
8. Transposing, $x=6+12=18$, = copper.
9. Subtracting, $24-18=6$, = silver.

Ex. (45.) $A=\dfrac{300((1.06)^7-1)}{.06(1.06)^7}.$

$$\begin{array}{rr} 1.06= & .025306\\ & 7\\ \hline & .177142\\ .06= & -2.778151\\ \hline & -2.955293\end{array}$$

$$\begin{array}{rr} 1.06= & .025306\\ & 7\\ \hline 1.50363= & .177142\\ 1 & \\ \hline .50363= & -1.702112\\ 300= & 2.477121\\ \hline & 2.179233\\ & -2.955293\\ \hline \$1674.714= & 3.223940\\ 300 & \\ \hline 1974.714. & \text{A's offer.}\end{array}$$

B's $=\dfrac{250((1.06)^7-1)}{.06(1.06)^7}.$

$$\begin{array}{rr} 1.06= & .025306\\ & 7\end{array}$$

$$\begin{array}{rr} 1.06= & .025306\\ & 7\\ \hline 1.50363= & .177142\\ 1 & \\ \hline .50363= & -1.702112\\ 250= & 2.397940\\ \hline \end{array}$$

$$\begin{array}{rr} & .177142 \\ .06= & -2.778151 \\ \hline & -2.955293 \end{array}$$

$$\begin{array}{lr} & 2.100052 \\ & =-2.955293 \\ \hline \$1395.595= & 3.144759 \\ 800 & \\ \hline 2195.595. & \text{B's offer.} \end{array}$$

$$\text{C's} = \frac{200((1.06)^7-1)}{.06(1.06)^7}.$$

$$\begin{array}{rr} 1.06= & .025306 \\ & 7 \\ \hline & .177142 \\ .06= & -2.778151 \\ \hline & -2.955293 \end{array}$$

$$\begin{array}{lr} 1.06= & .025306 \\ & 7 \\ \hline 1.50363= & .177142 \\ 1 & \\ \hline .50363= & -1.702112 \\ 200= & -2.301030 \\ \hline & 2.003142 \\ & -2.955293 \\ \hline \$1116.476= & 3.047849 \\ 1300 & \\ \hline 2416.476. & \text{C's offer.} \end{array}$$

A's, \$1974.714; B's, \$2195.595; C's, \$2416.476; D's, \$2500. D's offer is the best.

Ex. (46.) Let x = the age of the oldest son, and y = the youngest.

1. Then, by first condition, $(x+y)x=144$.
2. By second condition, $(x-y)y=14$.
3. Reducing (1), $x^2+xy=144$.
4. Reducing (2), $xy-y^2=14$.
5. Subtracting (4) from (3), $x^2+y^2=130$.
6. Transposing, $x^2=130-y^2$.
7. Evolving, $x=\sqrt{130-y^2}$.
8. Putting (7) into (4),
$$y\sqrt{130-y^2}-y^2=14.$$
9. Transposing, $y\sqrt{130-y^2}=14+y^2$.
10. Involving, $130y^2-y^4=196+28y^2+y^4$.
11. Transposing, &c., $2y^4-102y^2=-196$.
12. Dividing, $y^4-51y^2=-98$.

13. Comp. the sq., $y^4-51y^2+\frac{2601}{4}=-98+\frac{2601}{4}=\frac{2209}{4}$.
14. Evolving, $y^2-\frac{51}{2}=\frac{47}{2}$.
15. Transposing, $y^2=\frac{47}{2}+\frac{51}{2}=49$.
16. Evolving, $y=7$.
17. Substituting, $x=\sqrt{130-y^2}$.
18. Putting y into (17), $x=\sqrt{130-49}$.
19. Evolving, $x=\sqrt{81}=9$.

The ages of his sons were 9 and 7 years.

Ex. (47.) Let x and y = the two numbers.

1. Then, by first condition, $x+y=20$.
2. By second condition, $x^3+y^3=2060$.
3. Evolving (1), $x^2+2xy+y^2=400$.
4. Dividing (2) by (1), $x^2-xy+y^2=103$.
5. Subtracting (4) from (3), $3xy=297$.
6. Dividing, $xy=99$.
7. Dividing by y, $x=\frac{99}{y}$.
8. Putting x into (1), $\frac{99}{y}=20-y$.
9. Multiplying by y, $99=20y-y^2$.
10. Transposing, $y^2-20y=-99$.
11. Completing the square, $y^2-20y+100=-99+100=1$.
12. Evolving, $y-10=1$.
13. Transposing, $y=10+1=11$.
14. Putting y into (7), $x=\frac{99}{11}=9$.

Hence the numbers are 9 and 11.

Ex. (48.) Let x and y = the two numbers.

1. Then, by first condition, $x^2+xy=112$.
2. By second condition, $xy-y^2=12$.
3. Subtracting (2) from (1), $x^2+y^2=100$.

4. Transposing, $x^2=100-y^2$.
5. Evolving, $x=\sqrt{100-y^2}$.
6. Putting x into (1),

$$100-y^2+y\sqrt{100-y^2}=112.$$

7. Transposing, &c., $y\sqrt{100-y^2}=12+y^2$.
8. Evolving, $100y^2-y^4=144+24y^2+y^4$.
9. Transposing and reducing, $y^4-38y^2=-72$.
10. Completing the sq., $y^4-38y^2+361=361-72=289$.
11. Evolving, $y^2-19=17$.
12. Transposing, $y^2=17+19=36$.
13. Evolving, $y=6$.
14. Putting the value of y into (4), $x^2=100-36=64$.
15. Evolving, $x=8$.

Hence the numbers are 8 and 6.

Ex. (49.) Let $x =$ the number.

1. Then, by conditions, $x+2\sqrt{x}=24$.
2. Completing the square, $x+2\sqrt{x}+1=24+1=25$.
3. Evolving, $\sqrt{x}+1=5$.
4. Transposing, $\sqrt{x}=5-1=4$.
5. Involving, $x=16$.

Ex. (50.) It is evident that the answer to this question will be twice that of the 37th.

Therefore, 2×\$2.819748=\$5.639496.

Ex. (51.) We first find the solid contents of the plank.

$84\frac{1}{2}\times 144.\times 3=38504$ cubic inches.

If we suppose x to be the length of the box outside, then $x-6$ $=$ the length inside. And it is evident that the contents of the plank will be equal to the difference of the cubes of these two quantities.

Therefore, $(x-6)^3=(x^3-18x^2+108x-216)$ = contents inside.

And $x\times x\times x=x^3$ = contents of a block whose dimensions are equal to the box.

1. Then, $x^3-(x^3-18x^2+108x-216)=38504.$
2. Transposing and subtracting, $x^2-6x=2016.$
3. Completing the square, $x^2-6x+9=2016+9=2025.$
4. Evolving, $x-3=45.$
5. Transposing, $x=45+3=48$ inches.

Ex. (52.) $62\frac{25}{72}\times144\times2\frac{1}{2}=22445$ cubic inches in the plank. Let $x=$ the height and width of the box, and $4x=$ the length outside; therefore, $x-5$, $x-5$, and $4x-5$, will be the dimensions inside.

1. Hence, $x\times x\times4x-(x-5)(x-5)(4x-5)=22445.$
2. Multiplying, $4x^3-(4x^3-45x^2+150x-125)=22445.$
3. Subtracting, &c., $45x^2-150x=22445-125=22320.$
4. Dividing, &c., $x^2-\frac{10x}{3}=496.$
5. Comp. the square, $x^2-\frac{10x}{3}+\frac{25}{9}=496+\frac{25}{9}=\frac{4489}{9}.$
6. Evolving, $x-\frac{5}{3}=\frac{67}{3}.$
7. Transposing, $x=\frac{67}{3}+\frac{5}{3}=\frac{72}{3}=24$ inches,

the height and width of the box, and $24\times4=96$ inches, the length. That is, the length is $96\div12=8$ feet, and the height and width $24\div12=2$ feet.

Ex. (53.) Let $x=$ the quantity of wine drawn off at first. Therefore, $20-x=$ the quantity remaining, or the quantity of water in the second.

Then, $20 : x :: x : \frac{x^2}{20}=$ quantity of wine returned to the first, and $x-\frac{x^2}{20}$ will be the second.

Therefore, $20 : 20-x+\frac{x^2}{20} :: \frac{20}{3} : \frac{x^2-20x+400}{60}=$ the quantity of wine in $6\frac{2}{3}$ gallons.

1. Therefore,

$$\frac{20x-x^2}{20}+\frac{x^2-20x+400}{60}=\frac{2}{3}\left(\frac{x^2-20x+400}{20}\right).$$

2. Clearing of fractions and uniting terms, we have $60x-3x^2=x^2-20x+400.$
3. Transposing, &c., $4x^2-80x+400=0.$
4. Dividing, $x^2-20+100=0.$
5. Evolving, $x-10=0.$
6. Transposing, $x=10$ gallons.

Ex. (54.) As A has travelled 2¾ hours, at the rate of 4 miles an hour, it is evident he is 11 miles, or 44 quarter-miles, " ahead of B, who travels 2 quarters of a mile the first hour, 3 quarters the second hour, and so on, gaining 1 quarter of a mile each hour. We have, therefore, the first term 2, the common difference 1, and the sum of the series 44, to find the number of terms.

To obtain this, we have the following formula. See Algebra, page 231.

$$n=\frac{\sqrt{(2a-d)^2+8ds}\pm\overline{2a+d}}{2d}=\frac{\sqrt{(4-1)^2+352}+\overline{4-1}}{2}=8 \text{ hrs.}$$

Ex. (55.) Let x, xy, xy^2, xy^3 = the numbers.

1. By first condition, $x+xy=15.$
2. By second condition, $xy^2+xy^3=60.$
3. Separating into factors, $y^2(x+xy)=60.$
4. Putting (1) into (3), $15y^2=60.$
5. Dividing, $y^2=4.$
6. Evolving, $y=\pm2.$
7. Putting y into (1), $x+2x=3x=15.$
8. Dividing, $x=5.$

The numbers, therefore, are 5, $2\times5=10$, $5\times2^2=20$, $5\times2=40$.

Ex. (56.) Let $x+3y$, $x+y$, $x-y$, $x-3y$ = the numbers.

1. By first condition, $2x^2+18y^2=200.$
2. By second condition, $2x^2+2y^2=136.$
3. Subtracting (2) from (1), $16y^2=64.$
4. Evolving, $4y=\pm8.$
5. Dividing, $y=\pm2.$

6. Half of (2), $x^2+y^2=68.$
7. Putting y^2 into (6), $x^2+4=68.$
8. Transposing, $x^2=68-4=64.$
9. Evolving, $x=8.$

The numbers therefore are, $8+6=14$, $8+2=10$, $8-2=6$, $8-6=2$.

Ex. (57.) Let x = the number of yards.

Then $\frac{147}{x}=$ the buying price.

And $\frac{120\frac{1}{4}}{(x-12)}=$ the selling price.

1. Conditions, $\frac{147}{x}=\frac{120\frac{1}{4}}{(x-12)}-\frac{1}{4}.$
2. Clearing of fractions, $588x-7056=481-x^2+12x.$
3. Transposing and uniting, $x^2+95x=7056.$
4. Completing the square,

$$x^2+95x+\frac{9025}{4}=7056+\frac{9025}{4}=\frac{37249}{4}$$

5. Evolving, $x+\frac{95}{2}=\frac{193}{2}.$
6. Transposing, $x=\frac{193}{2}-\frac{95}{2}=49$, No. of yds.
7. Dividing, $147\div 49=$ £3 per yard.

Ex. (58.) Let x = the bushels of rye, and y = the bushels of wheat.

1. Then, by first condition, $x-y:y::100:x.$
2. By second condition, $x:x-y::y:4.$
3. By equality, $x:y::25y:x.$
4. Multiplying extremes, $x^2=25y^2.$
5. Evolving, $x=5y.$
6. Putting x into (2). $5y:4y::y:4.$
7. Cancelling, $5:1::y:1.$
8. Multiplying extremes, &c., $y=5.$
9. Substituting, $x=5y=25.$

There were 25 bushels of rye, and 5 bushels of wheat.

Ex. (59.) Let x^2 and y^2 be the two numbers.

1. By first condition, $x^2y=48$.
2. By second condition, $xy^2=36$.
3. Dividing (1) by x, $xy=\frac{48}{x}$.
4. Dividing (2) by y, $xy=\frac{36}{y}$.
5. Putting xy of (3) and (4) equal, $\frac{48}{x}=\frac{36}{y}$.
6. Dividing by 12, $\frac{4}{x}=\frac{3}{y}$.
7. Clearing of fractions, $4y=3x$.
8. Dividing, $y=\frac{3x}{4}$.
9. Putting y into (1), $\frac{3x^3}{4}=48$.
10. Clearing of fractions, &c., $x^3=64$.
11. Evolving, $x=4$.
12. Substituting, $y=\frac{3x}{4}=3$.

The numbers, therefore, are $4^2=16$, $3^2=9$.

Ex. (60.) Let x = the greater, and y = the less.

1. By first condition, $(x-y)\frac{x}{y}=48$.
2. By second condition, $(x-y)\frac{y}{x}=3$.
3. Dividing (1) by (2), $\frac{x^2}{y^2}=16$.
4. Evolving (3), $\frac{x}{y}=\pm4$.
5. Clearing of fractions, $x=\pm4y$.
6. Putting plus value of (4) and (5) into (1), $(4y-y)4=48$.
7. Multiplying, &c., $12y=48$.
8. Dividing, $y=4$.

9. Putting (8) into (5), $x=4\times4=16.$
10. Hence the values of x and y are 16 and 4.
11. But if $x=-4y,$
12. Then $x=-4\times4=-16.$
13. Putting the value of x into (4), $\frac{-16}{y}=-4.$
14. Clearing of fractions, &c., $y=4.$
15. Hence x and y may be —16, and 4.

Ex. (61.) Let x = the greater, and y = the less.

1. Then, by first condition, $x^2y=448.$
2. By second condition, $xy^2=392.$
3. Dividing (1) by x, $xy=\frac{448}{x}.$
4. Dividing (2) by y, $xy=\frac{392}{y}.$
5. Equality of (3) and (4), $\frac{448}{x}=\frac{392}{y}.$
6. Dividing, $\frac{8}{x}=\frac{7}{y}.$
7. Reducing terms, $x=\frac{8y}{7}.$
8. Putting x into (2), $\frac{8y^3}{7}=392.$
9. Dividing (8) by 8, $\frac{y^3}{7}=49.$
10. Multiplying, $y^3=392.$
11. Evolving, $y=7.$
12. Substituting, $x=\frac{7\times8}{7}=8.$

Ex. (62.) Let x and y be the numbers.

1. First condition, $\sqrt{27x}=27y.$
2. Second condition, $x=27y^2.$
3. Third condition, $\sqrt[3]{3x}=3y.$
4. Fourth condition, $x=9y^3.$

5. Value of x in (2) and (4), $9y^3=27y^2$.
6. Dividing by $9y^3$, $y=3$.
7. Substituting, $x=27y^2=243$.

The numbers are 243 and 3.

Ex. (63.) Let x = the side of the greater, and y = the side of the less.

1. First condition, $x^3-y^3=117$.
2. Second condition, $x-y=3$.
3. Cubing (2), $x^3-3x^2y+3xy^2-y^3=27$.
4. Subtracting (3) from (1), $3x^2y-3xy^2=90$.
5. Dividing into factors, &c., $xy(x-y)=30$.
6. Dividing (4) by $x-y$, &c., $3xy=30$.
7. Dividing by 3, $xy=10$.
8. Squaring (2), $x^2-2xy+y^2=9$.
9. Multiplying (7) by 4, $4xy=40$.
10. Adding (8) and (9), $x^2+2xy+y=49$.
11. Evolving, $x+y=\pm7$.
12. Adding (11) and (2), $2x=10$.
13. Dividing, $x=5$.
14. Subtracting (2) from (11), $2y=4$.
15. Dividing, $y=2$.

Hence the sides of the stacks are 5 and 2 yards, respectively.

Ex. (64.) $S=\frac{n}{2}(2a+(n-1).-d)=\frac{20}{2}(40+(20-1).-2)=$ 20 miles.

Ex. (65.) Let x = the longer side of the parallelogram, and y = the shorter side.

1. Then, by first condition, $x^2+y^2=60^2=3600$.
2. By second condition, $x : y :: 4 : 3$.
3. Multiplying extremes, &c., $3x=4y$.
4. Dividing, $x=\frac{4y}{3}$.
5. Putting y into (1), $\frac{16y^2}{9}+y^2=3600$.

6. Clearing of fractions, $25y^2=32400$.
7. Evolving, $5y=180$.
8. Dividing, $y=36$.
9. Substituting, $x=\frac{4y}{3}=48$.
10. Multiplying, $36\times48=1728$ sq. rds.
11. Dividing, $1728\div160=10$A. 3R. 8P.

Ex. (66.) Let x = the lady's age.

1. Then, by conditions, $\frac{x}{2}+\sqrt{x}-12=0$.
2. Multiplying, &c., $x+2\sqrt{x}=24$.
3. Comp. the square, $x+2\sqrt{x}+1=24+1=25$.
4. Evolving, $\sqrt{x}+1=5$.
5. Transposing, $\sqrt{x}=5-1=4$.
6. Evolving, $x=16$, lady's age.

Ex. (67.) Let x = the number.

1. Then, by conditions, $x+1 : x+7 :: x+7 : x+19$.
2. Multiplying extremes, &c., $x^2+20x+19=x^2+14x+49$.
3. Uniting terms, $6x=30$.
4. Dividing, $x=5$.

Ex. (68.) Let x and y = the two numbers.

1. Then, by first condition, $x+y=12$.
2. Transposing, $x=12-y$.
3. Second condition, $12-y : y :: (12-y)-y : 40$.
4. Multiplying extremes, $-12y+2y^2=480-40y$.
5. Dividing, $y^2-6y=240-20y$.
6. Transposing, $y^2+14y-=240$.
7. Comp. the sq., $y^2+14y+49=240+49=289$.
8. Evolving, $y+7=17$.
9. Transposing, $y=17-7=10$.
10. Subtracting, $x=12-10=2$.

Hence the two numbers are 2 and 10.

Ex. (69.) Let x = the larger number, and y = the less

1. First condition, $xy=54.$
2. Second condition, $x : y :: x+y : 10.$
3. Multiplying extremes, &c., $10x=xy+y^2.$
4. Dividing, $x=\frac{xy+y^2}{10}.$
5. Dividing (1) by y, $x=\frac{54}{y}.$
6. Putting x into (4), $\frac{54}{y}=\frac{xy+y^2}{10}.$
7. Substituting 54 for xy, $\frac{54}{y}=\frac{54+y^2}{10}.$
8. Clearing of fractions, $540=54y+y^3.$
9. Transposing, $y^3+54y=540=6\times90.$
10. Multiplying by y, $y^4+54y^2=540y.$
11. Transposing, $y^4=-54y^2+540y.$
12. Adding $90y^2$, $y^4+90y^2=90y^2-54y^2+540y.$
13. Collecting terms, $y^4+90y^2=36y^2+540y.$
14. Comp. the square, $y^4+90y^2+2025=36y^2+540y+2025$
15. Evolving, $y^2+45=6y+45.$
16. Cancelling, $y^2=6y.$
17. Dividing, $y=6.$
18. Substituting, $x=\frac{54}{6}=9.$

Hence the numbers are 9 and 6.

Ex. (70.) Let x = the larger part, and $20-x$ = the less.

1. Conditions, $x^2 : (20-x)^2 :: 9 : 4.$
2. Expanding, $x^2 : 400-40x+x^2 :: 9 : 4.$
3. Multiplying extremes, &c., $4x^2=3600-360x+9x^2.$
4. Transposing, $5x^2=360x-3600.$
5. Dividing, $x^2=72x-720.$
6. Transposing, $x^2-72x=-720.$
7. Comp. the square, $x^2-72x+1236=-720+1236=576.$
8. Evolving, $x-36=\pm24.$
9. Transposing, $x=\pm24+36=12.$
10. Subtracting, $20-12=8.$

Hence the two parts are 12 and 8.

Ex. (71.) Let x = the larger part, and $24-x$ = the less.

1. Conditions, $\frac{x}{24-x}:\frac{24-x}{x}::9:1.$
2. Multiplying extremes, &c., $\frac{x}{24-x}=\frac{216-9x}{x}.$
3. Clearing of fractions, $x^2=5184-432x+9x^2.$
4. Transposing, &c., $8x^2-432x=-5184.$
5. Dividing, $x^2-54x=-648.$
6. Comp. the square, $x^2-54x+729=-648+729=81.$
7. Evolving, $x-27=\pm9.$
8. Transposing, $x=\pm9+27=18.$
9. Subtracting, $24-18=6.$

Hence the numbers are 18 and 6.

Ex. (72.) Let x = the larger part, and $14-x$ = the less.

1. Conditions, $x^2:(14-x)^2::9:16.$
2. Expanding, $x^2:196-28x+x^2::9:16.$
3. Multiplying extremes, &c., $16x^2=1764-252x+9x^2.$
4. Dividing, $x^2+36x=252.$
5. Comp. the square, $x^2+36x+324=252+324=576.$
6. Evolving, $x+18=\pm24.$
7. Transposing, $x=\pm24-18=6.$
8. Subtracting, $14-6=8.$

Hence the parts are 6 and 8.

Ex. (73.) Let x = the larger part, and $12-x$ = the less.

1. Conditions, $x^2+\overline{12-x}^2:x^2-\overline{12-x}^2::5:3.$
2. Expanding,
   $x^2+144-24x+x^2:x^2-(144+24x+x^2)::5:3.$
3. Cancelling, $2x^2+144-24x:24x-144::5:3.$
4. Multiplying extremes, &c.,
   $6x^2-72x+432=120x-720.$
5. Reducing terms, $x^2-32x=-192.$
6. Comp. the square, $x^2-32x+256=-192+256=64.$
7. Evolving, $x-16=\pm8.$
8. Transposing, $x=\pm8+16=8.$
9. Dividing, $12-8=4.$

Hence the two parts are 4 and 8.

Ex. (74.) Let x and y represent the two numbers.

1. First condition, $xy=12.$
2. Second condition, $x^3+y^3 : (x+y)^3 :: 91 : 343.$
3. Expanding, $x^3+y^3 : x^3+3x^2y+3xy^2+y^3 :: 91 : 343.$
4. Dividing by 7, $x^3+y^3 : x^3+3x^2y+3xy^2+y^3 :: 13 : 49.$
5. Multiplying extremes, &c.,
   $$49(x^3+y^3)=13(x^3+3x^2y+3xy^2+y^3).$$
6. Dividing by $x+y$, $49(x^2-xy+y^2)=13(x^2+2xy+y^2).$
7. Multiplying, $49x^2-49xy+49y^2=13x^2+26xy+13y^2.$
8. Collecting terms, &c., $36x^2-75xy=-36y^2.$
9. Dividing by 36, $x^2-\frac{25xy}{12}=-y^2.$
10. Completing the square,
    $$x^2-\frac{25xy}{12}+\frac{625y^2}{576}=-y^2+\frac{625y^2}{576}=\frac{49y^2}{576}.$$
11. Evolving, $x-\frac{25y}{24}=\frac{7y}{24}.$
12. Transposing, $x=\frac{7y}{24}+\frac{25y}{24}=\frac{32y}{24}=\frac{4y}{3}.$
13. Dividing (1) by x, $x=\frac{12}{y}.$
14. Values of x in (12) and (13), $\frac{4y}{3}=\frac{12}{y}.$
15. Clearing of fractions, $4y^2=36.$
16. Dividing, $y^2=9.$
17. Evolving, $y=3.$
18. Substituting for y, $x=\frac{12}{3}=4.$

Hence the larger number is 4, and the smaller 3.

Ex. (75.) Let x = the larger number, and y = the less.

1. First condition, $xy=120.$
2. Second condition, $\overline{x+8}\times\overline{y+5}=300.$
3. Multiplying, $xy+8y+5x+40=300.$
4. Value of x in (1), $x=\frac{120}{y}.$

5. Putting xy into (3),

$$120+8y+5\left(\frac{120}{y}\right)+40=300.$$

6. Clearing of fractions,

$$120y+8y^2+600+40y=300y.$$

7. Collecting terms, $y^2-\frac{35y}{2}=-75.$

8. Completing the sq., $y^2-\frac{35y}{2}+\frac{1225}{16}=-75+\frac{1225}{16}=\frac{25}{16}.$

9. Evolving, $y-\frac{35}{4}=\frac{5}{4}.$

10. Transposing, $y=\frac{5}{4}+\frac{35}{4}=\frac{40}{4}=10.$

11. Dividing, $120\div10=12.$

Hence the larger is 12, and the less 10.

Ex. (76.) Let x = A's stock, then $x+60$ = B's stock, and $x-68$ = C's stock.

1. By conditions, $2x+60 : 2x-8 :: 5 : 4.$
2. Multiplying extremes, &c., $8x+240=10x-40.$
3. Collecting terms, $x=140$, A's stock.
4. Adding, $140+60=200$, B's "
5. Subtracting, $140-68=72$, C's "

Ex. (77.) Let x and y represent the stock of each respectively.

1. The first condition, $x+150 : y-50 :: 3 : 2.$
2. Multiplying extremes, &c., $2x+300=3y-150.$
3. Second condition, $x-50 : y+100 :: 5 : 9.$
4. Multiplying extremes, &c., $9x-450=5y+500.$
5. Multiplying (2) by 9, $18x+2700=27y-1350.$
6. Multiplying (4) by 2, $18x-900=10y+1000.$
7. Subtracting (6) from (5), $3600=17y-2350.$
8. Transposing, $17y=5950.$
9. Dividing, $y=350$, B's stock.
10. Putting y into (4), $9x-450=\overline{5\times350}+500.$
11. Uniting terms, $x=300$, A's stock.

Ex. (78.) If the first of the two numbers be x, it is evident that the other will be $\frac{7x}{5}$. And if the first of the second two numbers be y, the corresponding number will be $\frac{5y}{3}$.

1. Therefore, by first condition, $x+y : \frac{7x}{5}+\frac{5y}{3} :: 9 : 13.$
2. Multiplying extremes, &c., $13x+13y=\frac{63x}{5}+15y$
3. Clearing of fractions, $65x+65y=63x+75y.$
4. Uniting terms, $2x=10y.$
5. Dividing, $x=5y.$
6. By second condition, $\left(\frac{7x}{5}+\frac{5y}{3}\right)-(x+y)=16.$
7. Substituting, $7y+\frac{5y}{3}-(5y+y)=16.$
8. Multiplying by 3, $(21y+5y)-(15y+3y)=48.$
9. Collecting terms, $8y=48.$
10. Dividing, $y=6.$
11. Putting y into (5), $x=5\times6=30.$
12. Substituting for x and y, $5 : 7 :: 30 : 42.$
13. By conditions, $3 : 5 :: 6 : 10.$

Hence the first two numbers are 30 and 42, and the last two numbers are 6 and 10.

Ex. (79.) Let the proportion be $x : y$.

1. Then the cost of $x+y$ gallons $=10x+4y.$
2. And the selling price $=11x+11y.$
3. The gain will be $=x+7y.$
4. By conditions, $10x+4y : x+7y :: 100 : 43\frac{3}{4}.$
5. Multiplying extremes, &c., $43\frac{3}{4}(10x+4y)=100(x+7y).$
6. Dividing by $6\frac{1}{4}$, $7(10x+4y)=16(x+7y).$
7. Multiplying, $70x+28y=16x+112y.$
8. Dividing, $9x=14y.$

But if $9x=14y$, it is evident that the ratio of x to y will be as 14 to 9. There must therefore be 14 bushels of wheat to every 9 bushels of barley.

Ex. (80.) Let the three numbers sought be represented by x, y and z, respectively.

Then, as x days : a days : : 1, the whole work : $\frac{a}{x}$, the part performed by A in a days.

And as $y : a :: 1 : \frac{a}{y}$, the part performed by B in the same time.

Again, as $x : b :: 1 : \frac{b}{x}$, part performed by A in b days.

And as $z : b :: 1 : \frac{b}{z}$, part performed by C in b days.

Again, as $y : c :: 1 : \frac{c}{y}$, part performed by B in c days.

And as $z : c :: 1 : \frac{c}{z}$, part performed by C in c days.

Therefore, we have, by the question,

$$\frac{a}{x}+\frac{a}{y}=1, \text{ the whole work.}$$

$$\frac{b}{x}+\frac{b}{z}=1, \text{ the whole work.}$$

$$\frac{c}{y}+\frac{c}{z}=1, \text{ the whole work.}$$

Let the first of these equations be divided by a, the second by b, and the third by c, and we have

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{a}.$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b}.$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c}.$$

Add the above equations together, and divide their sum by 2, because each man's labor is reckoned twice, and we have $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}$. From this, if each of the three last equations be successively subtracted, we have

$$\frac{1}{z}=-\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}=\frac{-bc+ac+ab}{2abc}.$$

$$\frac{1}{y}=\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2c}=\frac{bc-ac+ab}{2abc}.$$

$$\frac{1}{x}=\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2c}=\frac{bc+ac-ab}{2abc}.$$

That is, $x=\frac{2abc}{bc+ac-ab}$ = the time A would dig the cellar.

$y=\frac{2abc}{ab+bc-ac}$ = the time B would dig the cellar.

$z=\frac{2abc}{ab+ac-bc}$ = the time C would dig the cellar.

And $\frac{2abc}{ab+ac+bc}$ = the time A, B, C, would dig the cellar

Ex. (81.) $153-45=108$; $108\div2=54$, A received. $54+45=99$, B received.

Let x = A's stock; then,

1. Conditions, $153 : 54 :: 833 : x$.
2. Multiplying, &c., $153x=54\times833=44982$.
3. Reducing terms, $x=\$294$, A's stock.
4. Subtracting, $833-294=\$539$, B's stock.

Ex. (82.) Let x = the lady's age, and y = the gentleman's

1. First condition, $x-4=\frac{y+4}{2}$.
2. Second condition, $x+4=y-4$.
3. Clearing (1) of fractions, $2x-8=y+4$.
4. Subtracting (2) from (3), $x-12=8$.
5. Transposing, $x=20$.
6. Substituting, $20+4=y-4$.
7. Transposing, &c., $y=28$.

Hence the lady's age was 20, and the gentleman's 28 years.

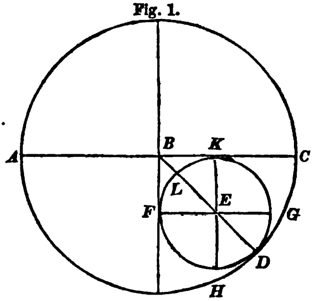
Ex. (83.) We have in this question a right-angled triangle, where the base and the sum of the perpendicular and hypothenuse are given to find the perpendicular. That is, we have the sum of two numbers and the difference of their squares given to find those numbers.

Let x = the longest side.

1. Then, by conditions, $\sqrt{x^2-(48-x)^2}=24$.
2. Evolving, $x^2-(2304-96x+x^2)=576$.
3. Subtracting, &c., $96x-2304=576$.
4. Reducing terms, $x=30$.
5. Subtracting, $48-30=18$ feet.

Ex. 84. Let ACD represent a circle containing the whole farm, and FLD a circle containing one of the son's farms. Assume the diameter of the son's farm to be 10 rods; then the contents will be $10\times10\times.785398=78.5398$ square rods. One-fourth part of this will be the contents of the quadrant FL, KE. That is, it will contain $78.5398\div4=19.634954$ square rods.

Fig. 1.



But the contents of the square $BKEF$ will be $5\times5=25$ square rods; therefore the contents of the figure $FBKL$ will be $25-19.634954=5.365046$ square rods. This figure, $FBKL$, will represent one-fourth part of the wife's farm. But, as her farm contained 160 square rods, this figure may be considered as containing one-fourth of her share, $160\div4=40$ square rods. And, as all similar figures are to each other as the squares of their homologous sides, therefore, as the contents of the assumed figure $FBKL$ is to the exact quantity which it should contain, so is the square of the semi-diameter of the assumed circle to the square of the semi-diameter of the required circle.

Therefore, as $5.365046 : 40 :: 5^2 : 186.39169$ = the square of FE or DE, and $\sqrt{186.39169}=13.652=FE$ or DE. As $BKEF$ is a square, and BE its diagonal, therefore BE will be equal to the square root of twice the square of FE.

$$13.652\times13.652\times2=372.754208.$$

$$\sqrt{372.754208}=19.306842=EB.$$
$$19.306842+13.652=32.958=BD,$$

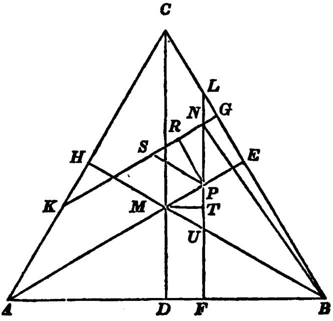
the semi-diameter of the circle *ACD*, and 2×32.958=65.916= its diameter. Therefore, 65.916×65.916×.7854=3412 poles, = 21 acres, 1 rood, 12 poles, the contents of the whole farm.

As the diameter of each of the son's farms is 2×13.652= 27.304 rods, the contents of each will be 27.304×27.304×.7854 =585½ square rods, = 3 acres, 2 roods, 25½ poles, or rods.

The four sons will, therefore, have 4×585½=2342 square rods. We now subtract what the sons and wife received from the contents of the whole farm, and one-fourth the remainder is what belongs to each daughter. 3412—(2342+160)=910; 910÷4=227½ poles, = 1 acre, 1 rood, 27½ poles. We therefore find the farm contained 21 acres, 1 rood, 12 poles.
Each son had 3 acres, 2 roods, 25½ poles.
Each daughter had 1 acre, 1 rood, 27½ poles.

Ex. (85.) Let *ABC* be an equilateral triangle, whose sides measure respectively 100 feet. Let the tower at *A* be 40 feet high, the one at *B* 45 feet, and the one at *C* 55 feet.

Fig. 2.



Bisect *AB* in *D*, *BC* in *E*, and *CA* in *H*. Draw *CD*, *BH*, and *AE*. The point *M* will be the centre of the triangle, and the lines *MA*, *MC* and *MB*, will be equal to each other. The lines *CD*, *BH* and *AE*, will each be $\sqrt{100^2-50^2}=86.60254$ feet. *MD* will be one-third of 86.60254 = 28.86751 feet.

As the tower *A* is 40 feet high, and the one at *B* 45 feet, we first ascertain the point *F*, which shall be equally distant from the top of the tower *A* and the top of the tower *B*. The tower *B* is 5 feet higher than *A*, and if half the height of *A* be added to half the height of *B*, the sum will be 42.5 feet.

Therefore, 100 : 5 : : 42.5 : 2.125 = the distance from

D to *F*. Let *F*, then, be a point in the *AB*, which is 2.125 feet from *D*, and is at equal distances from the top of the towers *A* and *B*.

AF will be 52.125 fect, and *FB* 47.875 feet.

Draw *FL* parallel to *DC*; any point in the line *FL* will be equally distant from the top of the towers *A* and *B*. By a process similar to the above we find the distance *EG* to be 5 feet. Thus, $100 : 10 :: 50 : 5$ feet $= EG$.

Draw *GK* parallel to *EA*; and, as the point *G* is equally distant from the top of the towers *B* and *C*, it is evident that, if any point be taken in the line *GK*, it will also be equally distant from the summit of said towers. Let this point be *N*. We have before shown that any point in the line *FL* will be equally distant from the top of the towers *A* and *B*. Let this point be *N*. Therefore the point *N* is equally distant from the summit of the three towers at *A*, *B* and *C*.

SPN is an equilateral triangle, and its perpendicular height *RP*, which is equal to *GE*, is 5 feet. The ratio of the side of an equilateral triangle to its perpendicular height is as 1 to .8660254. That is, $\sqrt{1-.25}=.8660254$.

Then, as $.8660254 : 1 :: RP : NP =$

$.8660254 : 1 :: 5 : 5.773502 = NP$.

Again, as $.8660254 : .5 :: 2.125 : 1.226869 = PT$.

$NP+PT+TF=5.773502+1.226869+28.86751=35.86788$ $=FN$.

As *FNB* is a right-angled triangle, *NB* will be equal to the square root of the sum of the squares of *FN* and *FB*. That is,

$$NB=\sqrt{(NF^2+FB^2)}=\sqrt{(35.867881^2+47.875^2)}=$$
$$\sqrt{(3578.52044069)}.$$

If to this we add the square of the height of the tower *B*, and extract the square root, we shall have the length of the ladder;

$$3578.52044069+(45^2=2025)=5603.52044066;$$

$\sqrt{(5603.52044066)}=74.856$ feet, the length of the ladder.

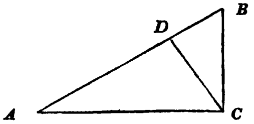
$5603.52044069-1600=4003.52044069=\sqrt{(4003.52044069)}$ $=63.2733$ feet $=$ the distance from the foot of the ladder to the base of the first tower. $5603.52044069-2025=3578.52044069$.

$\sqrt{(3578.52044069)}=59.820+$, distance from the base of the second tower; $5603.52044069-55^2=2578.52044069$; $\sqrt{(2578.52044069)}=50.779$ feet from the base of the third tower.

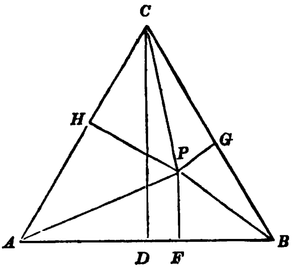
Thus, we find the distance from the foot of the ladder to the base of the first tower is 63.273+ feet; second tower, 59.820+ feet; third tower, 50.779+ feet; length of the ladder, 74.856+ feet.

Ex. (86.) Let $AB=c$; $AC=b$; $BC=a$; $x=AD$.

Fig. 3.



1. Therefore, $DB=c-x.$
2. Now, $CD^2=AC^2-AD^2=b^2-x^2.$
3. Also, $CD^2=CB^2-DB^2=a^2-(c-x)^2.$
4. Whence, $a^2-c^2+2cx-x^2=b^2-x^2.$
5. Therefore, $x=\dfrac{b^2+c^2-a^2}{2c}=AD.$
6. And $c-x=\dfrac{a^2+c^2-b^2}{2c}=BD.$

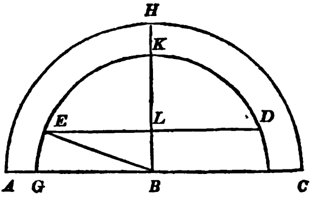


Ex. (87.) From the given point P let lines be drawn to the three angles of the triangle; and let CD be perpendicular to AB. Let $PF=a$; $PG=b$; $PH=c$; and $AD=x$. Then will $AC=AB=2x$; and $CD=\sqrt{AC^2-AD^2}=\sqrt{3x^2}=x\sqrt{3}$; and, consequently, the area of the whole triangle $ABC=CD\times AD=x^2\sqrt{3}$. But this triangle is composed of the three triangles APB, PBC, and APC; whereof the respective areas are ax, bx, and cx. Therefore, we have $x^2\sqrt{3}=ax+bx+cx$. Dividing both sides of the equation by x, we have $x\sqrt{3}=a+$

$b+c$; therefore $x=\frac{a+b+c}{\sqrt{3}}=\frac{20+30+36}{1.7320508}=49.652+$ feet. But $AB=2x=199.304$ feet. Therefore each side of the triangle will be 199.304+ feet. Ans.

Ex. (88.) $1^3+2^3+3^3+4^3+5^3=\frac{n^2(n+1)^2}{4}=\frac{5^2(5+1)^2}{4}=225.$
$225\times16=\$36000.$

Ex. (89.) Let AHC represent the loaf, and $EKDL$ the loaf after the crust is removed. Let LB or $HK=x$, the thickness of the crust; $AC=12$ inches, AB or $HB=6, KL=6-2x$. Draw BE; then BEL is a right-angled triangle, the right-angle being at L, the side $BL=x$, and $BE=6-x$. Therefore EL, the semi-base$=\sqrt{EB^2-LB^2}$, or $EL=\sqrt{\overline{6-x}^2-x^2}=\sqrt{36-12x}$, the semi-base of the loaf.



And by the rule for finding the segment of a sphere we have $(6\times6\times3+6\times6)6\times.5236=452.3904$, the contents of the large loaf. By dividing by 2, we have the contents of the smaller loaf$=226.1952$ cubic inches.

The following will therefore be the equation for the smaller loaf:

$$((36-12x)3+(6-2x)^2)\times(6-2x).5236=226.1952.$$

We divide this equation by .5236, and obtain

$$((36-12x)3)+(6-2x)^2)\times(6-2x)=432.$$

By dividing by 2 we have

$$((36-12x)3+(6-2x)^2)\times(3-x)=216.$$

By reduction, $x^3-18x^2+81x-54=0$.

We first suppose $x=.7$ or $.8$, and substituting these supposed values of x, we have $(.7)^3-18(.7^2)+81(.7)-54=0$.

Evolving, $.343-8.82+56.7-54=-5.777$, first error too small.

Second value of x, $(.8)-18(.8)^2+81(.8)-54=0$.

Evolving, $.512-11.52+64.8-54=.208$, second error too small.

Then $5.5 : .1 : : .2 : .003$, correction to be added to $.8$; thus, $.8+.003=.803$. This value of x we find too small, and we suppose $x = .803$ or $.804$.

Then by the supposition, $(.804)^3-18(.804)^2+81(.804)-54=0$.

Evolving, $.519718464-11.635488+65.124-54=+.00823$, second error too large.

Therefore, $.054 : .001 : : .004 : .00007407$, correction to be subtracted from $.804$; $.804-.00007407=.8039$, near the value of x. This value of x we find too large.

Again, suppose $x = .80387$, or $.80386$.

By supposition, $(.80387)^3-18\,(.80387)^2+81(.80387)-54=0$.

Expanding, $.519466402520603-11.6317255842+65.11347-54=+.00121$, too large.

2nd supposition, $(.80386)-18(.80386)^2+81(.80386)-54=0$.

Expanding, $.519447016552456-11.6314361928+65.11266-54=+.00067$, too large.

Therefore, $.00054 : .00001 : : .00067 : .00001$ to be subtracted from $.80386$; $.80386-.00001=.80385=x$, very near.

NOTE.—This answer is true to less than one hundred-thousandth part of an inch.

Ex. (90.) Let $AC=100$; $BD=80.27+$, or $\sqrt{6444\frac{4}{9}}$; $EF=32$, and $x=BA$, to find AB, BC, and DC.

By similarity of triangles, and the rules of proportion, we find,

$S : e : : m : a.$

$S : e : : o : z.$

$S : e : : w : z.$

$S : e : : w : r.$

$S : e : : o : r.$

$S : e : : c : y.$

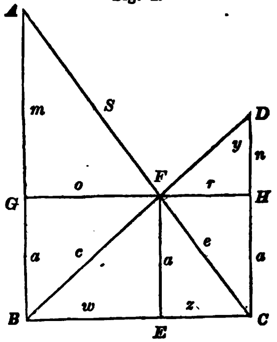
$S : e : : a : n.$

$S : e : : m : a.$

$S : e : : a+m : a+n.$

Or, $AG : FE : : FE : DH.$

Fig. 4.



Therefore, $x-32 : 32 :: 32 : \frac{1024}{x-32} = DH.$

$\overline{CA}^2 = 10000$; $a^2 = 32^2 = 1024$, therefore $mn = 1024$.

And $m = \frac{1024}{n}$, and $n = \frac{1024}{m}$.

$\overline{DB}^2 = 6444\frac{4}{9}$; therefore, $\overline{CA^2} - \overline{DB^2} = 10000 - 6444\frac{4}{9} = 3555\frac{5}{9}$.

And $\overline{32+m}^2 = \overline{32+n}^2 + 3555\frac{5}{9}$.

$$\left(32+\frac{1024}{n}\right)^2 = (32+n)^2 + 3555\tfrac{5}{9}.$$

Expanding, $1024 + \frac{65536}{n} + \frac{1048576}{n^2} = 1024 + 64n + n^2 + 3555\frac{5}{9}$.

$$\frac{65536}{n} + \frac{1048576}{n^2} = 64n + n^2 + 3555\tfrac{5}{9}.$$

$$65536n + 1048576 = 64n^3 + n^4 + 3555\tfrac{5}{9}n^2.$$

$$n^4 + 64n^3 + 3555\tfrac{5}{9}n^2 - 65536n = 1048576.$$

This equation is reduced by cubic and higher equation.

Let $n = 21$ and 22.

$$\begin{array}{rcrcr}
-1376256 & = & 65536n & = & -1441792 \\
1568000 & = & 3555\frac{5}{9}n^2 & = & 1720888\frac{8}{9} \\
592704 & = & 64n^3 & = & 681472 \\
194481 & = & n^4 & = & 234256 \\
\hline
978929 & & & & 1194824\frac{8}{9} \\
1048576 & & & & -1048576 \\
\hline
-69647 & & & & 146248\frac{8}{9}
\end{array}$$

$146248\frac{8}{9} + 69647 = 215895\frac{8}{9}$.

$215895\frac{8}{9} : 1 :: 69647 : .3+$.

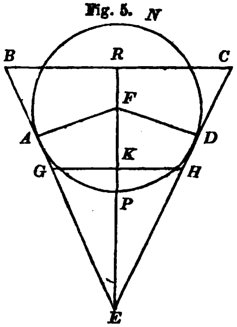
$n = 21 + .3 = 21\frac{1}{3} = DH$; $1024 \div 21\frac{1}{3} = 48 = AG$.

$32 + 21\frac{1}{3} = 53\frac{1}{3} = CD$; $32 + 48 = 80 = AB$.

$10000 - 6400 = 3600$; $\sqrt{3600} = 60$ BC.

Ex. (91.) Let *BCE* represent the conical glass, and *NAD* a ball put into it.

Fig. 5.



Let $BC=5$ inches, $BR=2\frac{1}{2}$, $RE=6$, and *AF* the semi-diameter of the ball = 2 inches.

Then $5\times5\times2\times.785398=39.2699=$ the cubic inches in the glass; $39.2699\div5=7.85398=$ the quantity of water in the glass.

Let $x=$ the axis of the ball that will be immersed in the water.

And $(\overline{3\times4}-\overline{2x\times x^2}).5236=6.2832x^2-1.0472x^3=$ contents of that part of the ball, *GK*, *HP*, that will be immersed in the water.

Then, by similarity of triangles, &c., we find,

1. $BR:RE::AF:AE=2\frac{1}{2}:6::2:4\frac{4}{5}=AE.$

2. $\overline{RE^2}+\overline{BR^2}=\overline{BE^2}$; $6.25+36=42.25$; $\sqrt{42.25}=6.5$ $=BE.$

3. $BR:BE::AF:FE$; $2\frac{1}{2}:6.5::2:5.2=FE.$

4. $FE-FP=PE$; $5.2-2=3.2=PE.$

Therefore, $KE=PE+PK=3.2+x.$

Then, as similar cones are to each other as the cubes of their altitude, we say,

5. As $6^3:39.2699::\overline{3.2+x}^3:7.85398+6.2832x^2$ $-1.0472x^3.$

Suppose $x=.54$; then,

6. $6^3:39.2699::\overline{3.2+x}^3=\overline{3.74}^3:7.85398+6.2832\times.54^2$ $-1.0472\times.54^3.$

7. Or, $216.:39.2699::52.313624:7.85398+1.83218112$ $-.16489.$

8. Collecting terms, $216:39.2699::52.313624:9.52127.$

9. But $216:39.2699::52.313624:9.51088.$

10. The result in (8.) should have been the same as in (9.).

11. Therefore, $9.52127-9.51088=.01039$ the first error too small.

Again, suppose $x=.55.$

12. Then 6^3 : 39.2699 : : $(3.2+.55)^3$: 9.585063.

13. Or, 216 : 39.2699 : : 52.734375 : 9.585063.

And, putting the last supposed value of x into (6), we have,

14. 216 : 39.2699 : : 52.734375 : $7.85398+6.2832(.55)^2 -1.0472(.55)^3$.

15. Or, 216 : 39.2699 : : 52.734375 : $7.85398+1.900668 -.1762279$.

16. Therefore, 216 : 39.2699 : : 52.734375 : 9.57842.

But, by the supposition, this last number should have been the same as in (12).

17. Therefore, $9.58506-9.57842=.00664+$.

We, therefore, find the last error to be .00664 too large.

First error, −.01039
Second error, +.00664

18. Sum of the errors, =.01703 : .01 : : .01039 : .006.

19. $.54+.006=.546$, *Answer.*

That is, the axis of the ball will be immersed $\frac{546}{1000}$ of an inch in the water.

Ex. (92.) To obtain the correct answer to this question, we place 144 balls on the bottom of the box, and on these we place 121 balls, and then 144, and so on; the layers of balls containing, alternately, 144 and 121. And when we have filled the box up to the 11th inch, we find we have

8 layers of balls, containing each 144=1152 balls.
and 7 layers, each containing 121=847 balls.

We find the number of layers in the eleven inches by dividing 11 by the square root of .50=.707+; thus, .11÷.707=15.

We now wish to find the number of balls that can be placed *above* the 11th inch. On the first 8 inches we can place 5 rows of 12 each = 60, and 4 rows of 11 each = 44. And on the last 4 inches we have sufficient space for 4 rows, each containing 12 balls = 48.

Hence, the box will contain 1152+847+60+44+48=2151 balls.

GREENLEAF'S

SERIES OF MATHEMATICS,

ADAPTED TO

ALL CLASSES OF PUPILS.

IN FIVE BOOKS.

BY BENJAMIN GREENLEAF, A. M.,

LATE PRINCIPAL OF BRADFORD (MASS.) TEACHERS' SEMINARY.

IMPROVED STEREOTYPE EDITIONS.

I. PRIMARY ARITHMETIC,

Or a System of Mental Arithmetic, with sufficient Slate Exercises added, to familiarize with written operations in the fundamental rules of the science, upon the Inductive Plan, for Primary and Intermediate Schools. Revised and enlarged edition, 144 pages, 18mo. Price 20 cts.

II. COMMON SCHOOL ARITHMETIC,

Or Introduction to the National Arithmetic, combining the Analytic and Synthetic methods, and forming a complete treatise for Common Schools. Improved stereotype edition, 324 pages, 12mo. Price 50 cts.

III. NATIONAL ARITHMETIC,

Being a thorough advanced course, intended particularly for Academies, High and Normal Schools. Improved edition, 360 pages, 12mo. Price 75 cts.

IV. PRACTICAL TREATISE ON ALGEBRA,

Designed for High Schools and Academies, and also for advanced students in Common Schools. Improved stereotype edition, 360 pages. Price 87½ cts.

V. PRACTICAL GEOMETRY,

Designed for High Schools and Academies, and for advanced classes in Common Schools. In preparation, and will soon be published.

COMPLETE KEYS

To the COMMON SCHOOL, NATIONAL ARITHMETIC, AND ALGEBRA, containing Solutions and Explanations, for Teachers only, (in three books.)

The above series has many important distinctive characteristics, which particularly recommend it to the attention of Teachers, School Directors, and others interested.

The ARRANGEMENT *of its several parts and subjects is lucid, progressive, and strictly philosophical.*

The RULES, DEFINITIONS, *and* ILLUSTRATIONS, *are expressed in language, simple, clear, concise, and accurate.*

The PROBLEMS *are of a practical nature, tending to interest the pupil, exercise his ingenuity, and secure useful mental discipline.*

It is a CONSECUTIVE SERIES, *graded to the wants of Primary, Intermediate, Grammar, and High Schools, Academies, and Normal Schools.*

The MECHANICAL EXECUTION *is neat and durable, an important consideration, too often disregarded.*

☞ The whole is the result of the ripe scholarship of a DISTINGUISHED PRACTICAL MATHEMATICIAN, who has had *more than thirty years' experience in teaching* pupils of all ages, grades, and capacities.

The PRIMARY ARITHMETIC constitutes, it is believed, the *best First Lessons* in numbers, now before the public. Being sufficiently comprehensive in its character, it is all the pupil needs, as a preparation for the next book in the series.

The COMMON SCHOOL ARITHMETIC contains *all the important rules of common Arithmetic with their practical applications*, and is ample to prepare the student for all ordinary business transactions. In its manner of treating *fractions*, and the prominence given through the entire book to *analysis*, and in its general adaptedness to the wants of the majority of schools, it is unsurpassed. It is a *complete system in itself*, though not so extensive as the National Arithmetic.

The NATIONAL ARITHMETIC contains a greater amount and variety of matter, strictly connected with the science, than will be found in any other treatise of the kind. It embraces a large amount of *mercantile* information, not usually included in works of this nature, but important to be possessed by all who are destined for the *warehouse* or *counting-room*. As a text-book for *advanced* and *normal classes* it has no equal.

The ALGEBRA furnishes what has been hitherto much desired, a thorough practical and theoretical text-book, suited to the wants of elementary schools, as well as academies, in a single volume of a portable form. Very comprehensive in its plan and details, and progressive in its gradation of problems, to the great satisfaction of teachers using it, it occupies the ground commonly given very inconveniently to two different books. Its several *demonstrations*, especially those connected with the *roots*, the *method of solving cubic equations by completing the square*, and the very complete *Table of Logarithms* at the end of the volume, are among its useful distinctive features.

The book has now been fully tested in the school-room, and the uniform testimony of teachers, who have made the trial, is, that its merit is fully equal to that of the arithmetics by the same author, and to which it proves the best and most appropriate sequel.

The PRACTICAL GEOMETRY, and other works, by Mr. Greenleaf, in his series, which are being prepared with great care, it is believed, will fully meet expectation, when published, and add still more to his high reputation as an author.

☞ Greenleaf's Arithmetics and Algebra are no untried books, or of doubtful reputation. No other works of the kind have, in the same time, secured so general an introduction, or been as highly commended by eminent mathematicians.

RECOMMENDATIONS OF GREENLEAF'S ARITHMETICS.

PROVIDENCE, July 28, 1855.

About five years ago, Greenleaf's "Common School Arithmetic" was [illegible] in the hands of the pupils of one class in each of our Grammar Schools, and, after a preliminary trial of nine months, we were so much pleased with it, that we unanimously recommended it as a suitable text-book. Our School Committee approved the choice, by then prescribing its general use, and since that time we have had the amplest means of testing its merits. Our experience has confirmed the wisdom of our selection, and now we can truly say, that, among all the books yet published, we know of no one which we would willingly adopt as its substitute.

Greenleaf's Primary Arithmetic was put into our Primary and Intermediate Schools three years since, and from what we learn from the teachers of those schools, as well as from what we ourselves have seen of its influence, we are of the opinion that it is eminently fitted for the place which it occupies.

Of the "National Arithmetic" it does not become us to speak with so much confidence, having never tested it as a text-book. We can only say that its merits are of so high an order as to have authorized its use in our High School, for six or seven years past.

(Signed),

ZUINGLIUS GROVER, *Principal of Prospect-st. Grammar School.*
ALBERT A. GAMWELL, " " *Fountain-st. Grammar School.*
LUCIUS A. WHEELOCK, " " *Elm-st. Grammar School.*
ALVAH W. GODDING, " " *Arnold-st. Grammar School.*
JAMES L. STONE, " " *Summer-st. Grammar School.*
CHARLES HUTCHINS, " " *Benefit-st. Grammar School.*

PROVIDENCE, August 20, 1855.

Having used several of the Arithmetics now before the public as text-books, in Grammar Schools, I have found none better fitted to accomplish the desired end than Greenleaf's Common School Arithmetic.

WILLIAM G. CROSBY, *Teacher of Providence High School.*

From John D. Philbrick, A. M., State Superintendent of Schools, and late Principal of the State Normal School, CONNECTICUT.

I have examined a copy of the improved edition of Greenleaf's School Arithmetic, which you had the kindness to send me, and I do not hesitate to assure you that I consider it an excellent arithmetic. Its work shows plainly the hand of experience. I think it now stands in the front rank of common school arithmetics. In explanations it has hit the true medium, being neither too abstruse nor too simple, doing enough for the learner, and yet leaving as much to him as is reasonable to expect him to do. It is neither too large nor too small, neither formidable nor deficient. It contains *all* that is necessary for all practical purposes. Among other good points, I was particularly pleased with its *oral* exercises, its progressive arrangement, the easy gradation and great diversity and ingenuity of its problems, and the perspicuity and precision of its rules and illustrations.

JOHN D. PHILBRICK.

QUINCY SCHOOL, Boston, March 5, 1849.

UNIVERSITY GRAMMAR SCHOOL, PROVIDENCE, Sept. 19, 1855.

BENJAMIN GREENLEAF, ESQ.: DEAR SIR, — We have used your National Arithmetic in our school during the last ten years, and now have no disposition to exchange it for any Arithmetic with which we are acquainted. It covers the whole ground of written Arithmetic, and it is so arranged that the pupil passes, by easy gradations, from simple exercises to the most complicated analysis. The principles are well stated the demonstrations clear and satisfactory, and the examples numerous and practical.

Respectfully yours,

MERRICK LYON, } *Principals.*
EMORY LYON, }

Extract from a letter from N. Tillinghast, A. M., Principal of the State Normal School, BRIDGEWATER, MASS.

I have used Mr. Greenleaf's National Arithmetic for several years in my school, and have seen no book that in the number and excellence of its examples has pleased me so well; and, so far as I have examined it, the revised edition seems superior in this respect, as in several others, to the former editions.

From Rev. H. S. Ramsdell, Chairman of the School Committee, THOMPSON, CONN.

THOMPSON, CT., February 22, 1855.

The Board of School Visitors of this town, after a careful examination of several arithmetics, adopted Greenleaf's Series, and put them into the eighteen schools under their supervision, more than a year since, to the exclusion of all others. Their use has more than realized their expectations. We not only find each book happily adapted to the place it was destined to fill, but so strictly *progressive*, so clear and precise in the statement of rules and principles, and in their illustration, and so ingenious and practical in the character of the examples, as exceedingly to interest the scholar, and to secure rapid progress in the science. After more than a year's experience in the use of the series, both teachers and scholars as well as the Board of Visitors, are free to express not only their satisfaction with the same, but their strong preference for them over any other series of the kind of which they have knowledge. We believe that their introduction into all the public schools of our state would be highly beneficial.

By order of the Board, H. S. RAMSDELL, *Chairman.*

From the Principal of Hulin's Seminary, CINCINNATI, OHIO.

CINCINNATI, March 18, 1853.

My experience satisfies me that Greenleaf's Arithmetic is fully deserving of the high reputation it has attained. Nearly two years' use of the book in my Seminary, has demonstrated to my mind its great superiority over any other book of the kind, in combining desirable mental discipline with a thorough practical knowledge of numbers. I regard it, in every respect, *a strong book*, and one likely to meet the wants of good teachers, generally. A. HULIN.

CAMBRIDGE, Ms., February 3, 1854.

Having made a thorough trial of Greenleaf's Common School Arithmetic, by carrying several classes through it, I am most happy to say that it has more than answered my highest expectations. In its arrangement, its illustrations, the clearness and conciseness of its rules, and, above all, in the fulness and variety of its examples, it is all that can be desired. With more than twenty years' experience in teaching Arithmetic from nearly all the treatises that have been in use during this period, I have found nothing that can compare with it. I have never been able to instruct from any book with so much ease and satisfaction to myself, or advantage to my pupils. It should supersede all other books upon the subject now in use. E. B. WHITMAN, *Principal of Hopkins' Grammar School.*

DAVENPORT, Iowa, September, 1855.

I have used Greenleaf's Series of Arithmetics, for several years, both in Massachusetts and in Iowa, with increasing confidence in their adaptation to convey a knowledge of numbers, and to produce energy and elasticity of intellect in those who study them.

So varied and numerous are the examples that it is *impossible* for the student to go through the series with any degree of thoroughness without having such a readiness in the use of numbers as to fit him to meet successfully all the ordinary demands of mathematical calculation.

The Algebra, by the same author, I have examined, and feel satisfied that it is well worthy of the reputation it has gained.

H. L. BULLEN, *Professor of Mathematics and Nat. Phil., in Iowa College.*

RECOMMENDATIONS OF GREENLEAF'S ALGEBRA:

PHILLIPS' ACADEMY, EXETER, N. H., February 26, 1855.

I have used, for several months, in the Mathematical Department of the Academy, Greenleaf's Algebra. It has stood, remarkably well, the test of the Recitation Room, — a test much severer than a private examination. Its problems, especially those in Quadratic Equations, are very much superior to those of most text-books in Algebra, which have come under my notice. These and the Miscellaneous Questions are well selected, and admirably suited to develop the ingenuity of the student, and familiarize him with the fundamental principles of the science. The discussion of the subject of Logarithms is rendered more valuable by the introduction of Logarithmic Tables at the end of the volume. On the whole I take pleasure in commending the book to teachers in High Schools and Academies.

J. G. HOYT, *Prof. of Mathematics.*

NEW ENGLAND NORMAL INSTITUTE, LANCASTER, MASS., June 13, 1855.

Greenleaf's Arithmetic and Algebra have been in use in the Mathematical Department of this Seminary, as the best text-books, in the judgment of our instructors, for the purpose of training the minds of teachers to their official duties in those branches. Our instructors in Mathematics have also regarded them as, on the whole, the best fitted of any of our class-books, in their department, for the general purposes of instruction in schools in which Arithmetic and Algebra are taught.

In the temporary absence of Professor HERMANN KRÜSI, of the Mathematical Department. WILLIAM RUSSELL, *Director.*

NEWBURYPORT, MASS., July, 10, 1853.

After a careful examination of Greenleaf's Algebra, we decided to introduce it into our school, and a trial of one term has fully confirmed the favorable impression we had received in regard to it. We are particularly pleased with the fitness of the examples to illustrate the principles, and fix them in the mind of the learner. We find some new matter, and such a combination and arrangement of the old, as will, in our opinion, render the work a very desirable text-book for the use of schools.

W. H. WELLS, *Principal of Putnam School.*

NEW HAVEN, January 2d, 1854.

I have used Greenleaf's Practical Treatise on Algebra in my school the past eight months, and have found it very convenient and valuable both to the teacher and pupil. It unfolds the science concisely, though with clearness. The distinguishing characteristic of the work, however, is seen in the great variety of practical examples and problems under the rules. I would, with confidence, commend the book to the attention of teachers of elementary mathematics, as one of the best that have been published in this country.

STILES FRENCH, *Teacher of Mathematics.*

STATE NORMAL SCHOOL, NEW BRITAIN, CT., APRIL 25, 1855.

Having carefully examined Greenleaf's National Arithmetic and Algebra, and used them in classes, I am able to speak of them in terms of high commendation. DAVID N. CAMP, *Prof. of Mathematics.*

SPRINGFIELD, MASS., June 4, 1855.

During the year past, Greenleaf's Algebra has been in use in this school, and, as a mathematical text-book, has been found very satisfactory. In copiousness and variety of problems, I do not know its superior; and they are so arranged that while the youngest pupil may comprehend and perform a large proportion, the most advanced will find those that will fully tax his mathematical powers. The subjects are clearly stated, and some of them are more completely demonstrated than in any other work I have used. I esteem the work very highly. A. PARISH, *Principal of the High School.*

State Normal School, Westfield, Mass., September 15, 1855.

Greenleaf's Algebra and National Arithmetic are now used as text-books in this institution. W. H. WELLS, *Principal.*

Quincy, Mass., October 25, 1854.

Having carefully examined Greenleaf's Algebra, and having made a thorough trial of it as a text-book in my school, I can confidently say that I regard it as one of the very best works of the kind now in use. It is peculiarly adapted to the purpose for which it is intended.

A. WELLINGTON, *Principal of the High School.*

Norwich, Ct., Oct. 22, 1854.

It is with pleasure I add my testimony to that of others in favor of Greenleaf's Algebra. The very name of the author of the National Arithmetic is sufficient guaranty for its excellence. In almost every respect it may be pronounced the best Algebra yet published. The principles are clearly stated, and the examples to illustrate and impress them sufficient, and selected with the same regard to their practical bearing that mark those in the author's arithmetic. I most cheerfully recommend the work to those who are in search of a thorough work upon the science of Algebra. JAMES LANE,

Principal of Norwich Grammar School.

Waltham, Mass., Sept. 28, 1853.

I think Greenleaf's Algebra, revised edition, better adapted to the present demand in our High Schools than any other work of the kind I have examined.

L. P. FROST, *Principal of High School.*

Westfield, Mass., March 2, 1855.

Greenleaf's Algebra has been in use in the Academy under my care for several terms. We displaced Robinson's Algebra to give it room, and think the change a good one. More and more acquaintance only confirms us in this idea.

The book is simple, plain, and practical. It is not too bulky or too learned. Most pupils can complete it in one or two terms; and yet it contains Algebra enough for academical use. It is very appropriate for the class of pupils that with us are passing from arithmetic into this field. We need only add, we like it. W. C. GOLDTHWAIT, *Principal of the Academy.*

Smithville Seminary, N. Scituate, R. I., February 12, 1855.

In compliance with your request, I now express an opinion concerning Greenleaf's Algebra. I had long felt the necessity of a better text-book on the science of Algebra. The work is before me. I am using Greenleaf's Algebra, and admire the ingenuity and correctness of its reasonings, and the tact of its operations. I would not exchange it for any other work of the kind with which I am acquainted.

H. M. WALLACE, *Prof. of Mathematics.*

Greenfield, Mass., September 18, 1854.

Greenleaf's Algebra was introduced into our school the last term. So far as our practical acquaintance with the work extends, we are much pleased with it. There seems to be a completeness without superfluity, or encumbrance from observations, which may well be left to the judgment of the pupil.

LUTHER B. LINCOLN, *Principal of the Public High School.*

Thetford Academy, October 14, 1853.

I have drilled a large class of students, during the term, in the new edition of Greenleaf's Algebra, as an experiment, and am happy to say, with great success. I recommend no book that has not stood the ordeal of my school-room. This book teaches well, and therefore has my approbation, and I shall continue to use it in school. Greenleaf's Algebra is eminently a *practical* book, and deserves the favor of every friend of sound learning.

HIRAM ORCUTT, *Principal of Thetford (Vt.) Academy.*

DEDHAM, MASS., Jan. 14, 1854.

The introduction of Greenleaf's Algebra into my school has convinced me of its great excellence. It does not aim to remove all difficulties from the student's path, but rather puts into his hands the best implements and rules for his work, and then confidently brings him in contact with all the obstacles with which he will be liable to meet in this branch of mathematics. I think it will be no easy matter to obstruct, with an algebraic difficulty, the path of one who has solved all the problems in this work. Truly, both the author and publishers of this book deserve the thanks of the friends of mathematical learning.

C. SLAFTER, *Principal of Public High School*.

WEST ROXBURY, Mass., September 29, 1854.

A careful examination of the revised edition of Greenleaf's Algebra has produced a very favorable impression on my mind, in regard to the merits of the work. Though not so extended and comprehensive as some more elaborate treatises on Algebra, it embraces as much matter as can well receive attention in the majority of our High Schools and Academies; and it seems to me remarkably well adapted to the purposes for which it was designed. The topics of which it treats are presented in a practical manner, and its examples are well suited to illustrate the principles they involve. Entertaining such confidence in this work as to lead me to adopt it as a text-book in my school, I cheerfully recommend it to the favorable notice of my fellow-teachers. D. B. HAGAR, *Principal of Elliot High School.*

Copy of a letter to the Publishers, from Rev. ELIAS NASON, *formerly Principal of the High School, Newburyport, Mass.*

NATICK, October 12, 1854.

I have spent some time, of late, in going through, article by article, your last edition of Greenleaf's Algebra ; and I find it to be what I expected, from my knowledge of the author's previous works, to find it : *a masterly production,* in which the elements of this beautiful science are ingeniously, clearly and naturally unfolded and explained ; in which the whole system, in all its parts and bearings and dependencies, is fitly joined together, and thus most orderly and harmoniously set before the learner's mind. The definitions are concise, exact and clear, the methods of solution easy and direct, and the examples and illustrations are, for the most part, original and to the point.

I like the book in all its parts — its matter, method, style and typographic execution; and I most heartily thank the author and publisher for making this valuable contribution to our school literature. Mr. Greenleaf is an exact and thorough mathematician, and hence whoever follows him will be sure to come out right in the end. It takes some thinking, I admit, to keep along with him, but then he leaves you at the last with power to go alone, and in the right direction also.

For his mathematical labors, to use a phrase of Cicero, "*he deserves well of his country.*" It was a great and noble undertaking to give the public an Arithmetic, which, after a fiery trial of many years, now fairly bears the title of the BEST; but it was a greater and nobler undertaking to produce an Algebra, which those who use it must pronounce superior to that book. ELIAS NASON.

I have examined the new edition of Greenleaf's Algebra very carefully, and have tested it quite thoroughly in the school-room. It seems to me that the work has been vastly improved by the revision which it has undergone. The correction of the errors which marred the former edition, and the introduction of a large amount of valuable matter, have added much to its merit as a text-book.

It now contains, in comparatively small compass, a variety of principles and processes, and a fuller and better collection of examples than I have seen in any similar work. Mr. Greenleaf has certainly succeeded in his aim to give the treatise "a practical character," and I cannot doubt that it will fully sustain the high reputation as an author which he has already gained by his Arithmetics.

W. J. ROLFE, *Master of Dorchester (Ms.) High School.*

PROGRESSIVE EXERCISES
IN
ENGLISH COMPOSITION.

BY RICHARD G. PARKER, A.M.

Improved Stereotype Edition. 144 pages, 12mo. Price $3.00 a dozen.

THE acknowledged superiority of this work, as an aid in teaching efficiently an important branch of education, has secured its introduction into many public schools, and into almost every Academy and higher Seminary throughout this country, besides being much used in the British Provinces and in England.

The School Committee of BOSTON authorized its introduction into the public schools of the city soon after the first edition was issued, to be the only work on Composition used in them.

It is about the only American school-book that has been republished and extensively used in the British Empire. In the United States alone, the extraordinary number of more than *one hundred and eighty thousand* have been sold. No better evidence can be given than this of its value as a text-book.

☞ *Teachers and others ordering this manual, to prevent mistakes, should be careful to give its distinctive title, "Parker's* EXERCISES *in Composition."*

From a large number of notices, from the most respectable sources, are selected the following

RECOMMENDATIONS.

From J. W. BULKLEY, A.M., *now Superintendent of Schools, Brooklyn, N. Y.*

I have examined "Parker's Exercises in Composition," and am delighted with the work. I have often felt the want of just that kind of aid that is here afforded. The use of this book will diminish the labor of the teacher, and greatly facilitate the progress of the pupil in a study that has hitherto been attended with many trials to the teacher, and perplexities to the learner.

If Mr. Parker has not strewed the path of the student with flowers, he has "removed many stumbling-blocks out of the way, made crooked things straight, and rough places smooth." It is certainly one of the happiest efforts that I have ever seen in this department of letters,—affording to the student a beautiful introduction to the most important principles and rules of rhetoric; and I would add, that, if carefully studied, it will afford a "*sure guide*" to written composition. I shall use my influence to secure its introduction to all our schools.

From SAMUEL P. NEWMAN, *Professor of Rhetoric in Bowdoin College.*

I have examined "Progressive Exercises in English Composition," by R. G. Parker, with some care, and hesitate not to express an opinion that it is well adapted to the purpose for which it is designed. It is well fitted to call into exercise the ingenuity of the pupil, to acquaint him with the more important principles and rules of rhetoric, and to guide and aid his first attempts in the difficult work of composition.

From MR. C. K. DILLAWAY, *Principal of the Latin School,* BOSTON.

Their clearness and simplicity strongly recommend them to the instructors in this important branch of education.

POPULAR ELOCUTIONARY WORK.

The National Speaker,

Designed for COMMON SCHOOLS and ACADEMIES. By HENRY B. MAGLATHLIN, A. M., Author of Practical Elocutionist. *Fifteenth Edition, revised* and *enlarged.* 360 pages, 12mo. Price, 87½ cts.

This work includes a *Complete Elocutionary Analysis*, as an introduction to a copious collection of new and interesting exercises, original and selected, in *Prose*, *Poetry*, and *Dialogue*, for *Declamation*, *Recitation*, or *Reading.* The selections are mostly *new*, and in accordance with the *Spirit of the Times.* Many of the Dialogues, and some of the other pieces, have never before been published; a majority of those not original appear in a work of this kind for the first time.

It contains exercises suited to pupils of both sexes, and of different ages. A part of the book is devoted to *addresses* and *exercises* especially adapted to *examination days*, *evening* and *day exhibitions*, and *presentation occasions*,— an original feature, adding much to its value. Exhibiting in its lessons every desirable variety of style, it may be used not only as a Speaker, but to great advantage as a "*Rhetorical Reader.*"

☞ The *extensive adoption* of this book in the best schools in various parts of the country, and the great satisfaction it gives on trial, is ample evidence of its superior merit.

The work is used in the *Public High Schools* of BOSTON, CAMBRIDGE, CHARLESTOWN, LOWELL, SPRINGFIELD, NEW BEDFORD, and in other important schools in MASSACHUSETTS; also in *State Normal School*, CT.; *Maine Wesleyan Seminary*, ME.; *Phillips Academy*, N. H.; *Collegiate Institute*, NEW HAVEN, CT.; *Collegiate Institute*, POUGHKEEPSIE, N. Y.; *Treemont Institute*, NORRISTOWN, PA.; *State University*, DEL.; *St. Timothy's Hall*, near BALTIMORE; *Union University*, TENN.; *Acadia College*, NOVA SCOTIA; and other institutions of high character.

No school should be without the National Speaker. *Distinguished Educationists recommend its use.* Among numerous others, have been received the following

TESTIMONIALS.

From JOHN D. PHILBRICK, A.M., *State Superintendent of Common Schools, and late Principal of State Normal School, Ct,*

The National Speaker is an excellent school-book. The compend of the principles of elocution, which it contains, is admirable,— being, at once, clear, comprehensive, and practical. It works well in the school-room. Its selections for declamation have been made with judgment and taste; and embrace as great a variety of style and matter as is desirable in such a work.

From C. C. CHASE, A. M., *Principal of Public High School, Lowell, Mass.*

I know of no work on the subject of Elocution which I can more highly recommend.

PARSING BOOK.

Class Book of Prose and Poetry,

Consisting of selections from the best English and American authors, designed as Exercises in Parsing, for the use of Common Schools and Academies. By Truman Rickard, A.M., and Hiram Orcutt, A.M., Principal of North Granville (N. Y.) Female Seminary. Revised and enlarged edition. 144 pages, 18mo. Price 20 cents.

The very extensive sale of this little volume, and its general acceptance with teachers, have induced the compilers to make still further efforts for its improvement. The body of the book remains unchanged, while there will be found an addition of twenty-four pages of important matter. It was designed that the Tables should be used in the analysis of sentences. But very few teachers, however, have used them at all, obviously for the reason that they were not understood. It has been the leading object, therefore, in preparing this enlarged edition, to arrange the principles of Syntax and Analysis in a systematic form, and to illustrate them fully by examples, so that both teachers and scholars may use the book understandingly.

Very important assistance has been rendered by Professor Alpheus Crosby, whose ripe scholarship and profound knowledge of the principles of general Grammar, render his suggestions of great value.

RECOMMENDATIONS.

From W. H. Wells, A.M., Author of "A Grammar of the English Language," and now Principal of the State Normal School, Westfield, Mass.

I have examined the "Class Book of Prose and Poetry," compiled by Messrs. Rickard & Orcutt, and take pleasure in expressing my unqualified approval of the plan and execution of the work. The first edition of the "Selections" was introduced as a class book in this institution about a year since, and the experiment has fully confirmed the favorable opinion which I then formed respecting its merits. The present edition is greatly improved, and cannot fail to meet with general favor.

W. H. WELLS,
Instructor English Department, Phillips' Academy, Andover.

From the author of "Gradual Lessons in Grammar," "Intellectual Algebra," &c.

I have examined with interest a little volume of Selections, to be used as a text-book in schools, for exercises in the analysis of language. The extracts are from standard writers, and offer a variety of examples to illustrate all the principles of grammar. The taste and judgment of the gentlemen who have compiled the work are sufficient evidence that it is well adapted to the purposes for which it is designed.

D. B. TOWER.

Extract of a letter from B. Greenleaf, Esq., Principal of Bradford Teachers' Seminary, and Author of the "Popular Series of Arithmetics."

The extracts are made with good taste and judgment, from the most approved authors, which, in connection with the Tables for Parsing, make it very valuable as a text-book. I think such a work has long been needed in our academies and high schools. I have introduced it into my seminary, believing the work superior to any other for parsing, that I have examined.

BENJAMIN GREENLEAF.

APPROVED CLASSICAL BOOKS.

FISK'S GREEK GRAMMAR.

A GRAMMAR OF THE GREEK LANGUAGE. By BENJAMIN FRANKLIN FISK. Forty-fifth stereotype edition.

FISK'S GREEK EXERCISES.

GREEK EXERCISES; containing the substance of the Greek Syntax, illustrated by Passages from the best Greek Authors, to be written out from the words given in their simplest form. By BENJAMIN FRANKLIN FISK. "Consuetudo et exercitatio facilitatem maxime parit."—Quintil. Adapted to the author's "Greek Grammar." Sixteenth stereotype edition.

Fisk's Greek Exercises are well adapted to illustrate the rules of the Grammar, and constitute a very useful accompaniment thereto.

From FORREST AND WYCKOFF, *Principals of Collegiate School, New York City.*

The requisites in a Manual of Grammar are simplicity and lucidness of arrangement, condensation of thought, and accuracy of principle and expression. These requisites Mr. Fisk appears to have attained in a considerable degree in his Greek Grammar, of which we have expressed approbation by introducing it into our school.

From BENJAMIN GREENLEAF, A. M., *Author of the "National Arithmetic," etc.*

For several years past, I have used Fisk's Greek Grammar in my seminary. I consider it a work of superior merit. It is well arranged, and the rules are clear and perspicuous. It is, in my opinion, better adapted to initiate pupils into the idiom of the Greek language, than any other treatise of the kind that I have examined. FISK'S GREEK EXERCISES should be used in connection with the Grammar. A work of this kind has long been needed. It is a production of great merit.

LEVERETT'S CÆSAR'S COMMENTARIES.

CAII JULII CÆSARIS COMMENTARII DE BELLO GALLICO AD CODICES PARISINOS RECENSITI, a N. L. Achaintre et N. E. Lemaire. Accesserunt Notulæ Anglicæ, atque Index Historicus et Geographicus. Curavit F. P. LEVERETT. Editio stereotypa.

From PROF. JOHN J. OWEN, *Editor of a Popular Classical Series.*

I have examined with some attention Cæsar's Commentaries, edited by Leverett, and Cicero's Orations, edited by Folsom, and am happy to recommend them to classical teachers, as being, in my estimation, far superior to any other editions of those works, to which students in this country have general access. The typography is fair and accurate, and the general appearance of the books does honor to the enterprising publisher. I hope these editions will be extensively used in our Academies and High Schools.

FOLSOM'S CICERO'S ORATIONS.

M. T. CICERONIS ORATIONES QUÆDAM SELECTÆ, Notis illustratæ. [By CHARLES FOLSOM, A.M.] In Usum Academiæ Exoniensis. Editio stereotypa, Tabulis Analyticis instructa.

From CHARLES E. WEST, A. M., *Principal of Rutgers Female Institute, New York.*

I take pleasure in commending to teachers the recent beautiful edition of Folsom's Cicero. The attractiveness of its text, notes, synoptical and analytical tables, and typographical execution, led me to place it in the hands of a class of young ladies, who are reading it with delight.

SCHOOL BOOK DEPOSITORY.

ROBERT S. DAVIS & CO.,

PUBLISHERS AND BOOKSELLERS,

118 WASHINGTON ST., BOSTON,

Have for Sale, in addition to their own Publications, an assortment of

APPROVED SCHOOL TEXT-BOOKS,

Used in the Common Schools, Academies and Colleges, in the United States and the British Provinces, together with a great variety of Books in the different departments of Literature and Science. Also, the best

SCHOOL STATIONERY,

INCLUDING THE VARIOUS SIZES AND QUALITIES OF

WRITING PAPERS, RULED AND PLAIN;

Steel Pens, Ink, Wafers, Sealing-wax, Lead and Slate Pencils, Slates, Writing, or Copy Books, Memorandum Books, and Blank Account Books.

☞ Teachers, School Directors, Booksellers and Country Merchants, supplied at the lowest market prices, wholesale and retail.

All orders promptly and faithfully attended to, if accompanied by a cash remittance, or a suitable reference.

☞ *Specimens of either of the Books described in this Catalogue will be sent, by mail or otherwise,* POSTAGE PAID, *to Teachers and School Directors, for examination, on the Publishers' receiving the amount of the printed prices, in postage stamps, with explicit directions for forwarding the Books.*

The Books will also be furnished, FOR A FIRST INTRODUCTION, *in exchange for others in use,* AT ONE HALF THE RETAIL PRICES.

It will be observed that the Books are all handsomely printed on white paper, neatly and durably bound, and are sold at low prices.

All interested are invited to correspond with us on the subject.

ROBERT S. DAVIS & CO., School Book Publishers,

No. 118 Washington St., Boston.

10-4